DJM3C - MECHANICS

Unit I: Forces acting at a point – Parallelogram of forces – triangle of forces – Lami's Theorem, Parallel forces and moments – Couples – Equilibrium of three forces acting on a rigid body.

Unit II: Friction – Laws of friction – equilibrium of a particle (i) on a rough inclined plane, (ii) under a force parallel to the plane , (iii) under any force – equilibrium of strings – equation of the common catenary – tension at any point – geometrical properties of common catenary – uniform chain under the action of gravity – suspension bridge.

Unit III: Dynamics – projectiles – equation of path, range etc – range on an inclined plane – motion on an inclined plane. Impulsive forces – collision of elastic bodies – laws of impact – direct and oblique impact – impact on a fixed plane.

Unit IV: Simple harmonic motion in a straight line – geometrical representation – composition of SHMS of the same period in the same lane and along two perpendicular directions – particles suspended by spring – SHM on a curve – simple pendulum – simple equivalent pendulum – second's pendulum.

Unit V: Motion under the action of central forces – velocity and acceleration in polar co-ordinates – differential equation of central orbit – pedal equation of central orbit – apses – apsidal distances – inverse square law.

Reference Books:

1. Statics and Dynamics: S. Narayanan
2. Statics and Dynamics: M.K. Venkataraman
3. Statics : Manickavachagompillai
4. Dynamics : Duraipandian

Introduction

"***Mathematics is the Queen of the Sciences and Number Theory is the Queen of Mathematics***" - Gauss.

Mechanics is a branch of Science which deals with the action of forces on bodies. Mechanics has two branches called Statics and Dynamics.

Statics is the branch of Mechanics which deals with bodies remain at rest under the influence of forces.

Dynamics is the branch of Mechanics which deals with bodies in motion under the action of forces.

Definitions:

Space: The region where various events take place is called a space.

Body: A portion of a matter is called a body.
Rigid body: A body consists of innumerable particles in which the distance between any two particles remains the same in all positions of the body is called a rigid body.

Particle: A particle is a body which is very small whose position at any time coincides with a point.

Motion: If a body changes its position under the action of forces, then it is said to be in motion.

Path of a particle: It is the curve joining the different positions of the particle in space while in motion.

Speed: The rate at which the body describes its path. It is a scalar quantity.

Displacement (vector quantity): It is the change in the positions of a particle in a certain interval.
Velocity (vector quantity): It is the rate of change of displacement.

Acceleration (vector quantity): It is the rate of change of velocity.
Equilibrium: A body at rest under the action of any number of forces on it is said to be in equilibrium.
Equilibrium of two forces

$Q \longleftarrow\!\!\!\longrightarrow P$

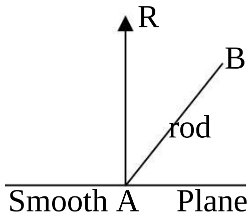
If two forces P, Q act on a body such that they have equal magnitude, opposite directions, same line of action then they are in equilibrium.

Force (vector): Force is any cause which produces or tends to produce a change in the existing state of rest of a body or of its uniform motion in a straight line. Force is represented by a straight line (through the point of application) which has both magnitude and direction.

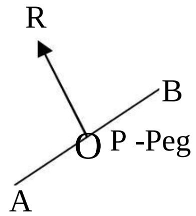
Types of forces: Weight, attraction, repulsion, tension, thrust, friction etc.
By Newton's third law, action and reaction are always equal and opposite.

Directions of Normal Reaction 'R' at the point of contact.

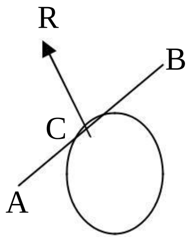
1. When a rod AB is in contact with a smooth plane, R is perpendicular to the plane at the point of contact A.



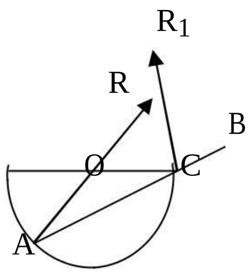
2. When a rod AB is resting on a smooth peg P, R is perpendicular to the rod at the point of contact P.



3. When a rod AB is resting on a smooth sphere, R is normal to the sphere at the point of contact C.



4. When a rod AB is resting on the rim of a hemisphere, with one end A in contact with the inner surface and C in contact with the rim. Then the normal reactions R at A is normal to the spherical surface and passes through the centre O, R_1 at C is perpendicular to the rod.



Regular polygon is a polygon with equal sides. Its vertices lie on a circle.

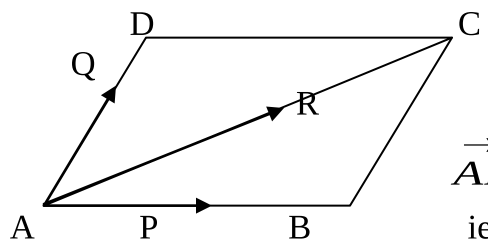
UNIT I
Forces Acting at a Point

Introduction

Forces are represented by straight lines with magnitude and direction. Forces acting on a rigid body may be represented by straight lines with magnitude and direction passing through the same point and we say the forces are acting at a point. If P_1, P_2, P_3 …….. are the forces acting on a rigid body it is easy to find a single force whose effect is same as the combined effect of P_1, P_2, P_3 …….. Then the single force is called the resultant. P_1, P_2, P_3 ….. are called the components of the resultant. In this section we study some theorems and methods to find the resultant of two or more forces acting at a point.

1.1 Parallelogram law of forces (Fundamental theorem in statics)

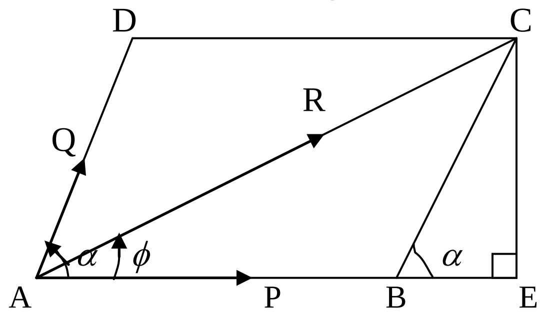
If two forces acting at a point be represented in magnitude and direction by the sides of a parallelogram drawn from the point, their resultant is represented both in magnitude and direction by the diagonal of the parallelogram drawn through that point.



$$\vec{AB} + \vec{AD} = \vec{AC}$$

ie) P + Q = R

The resultant of two forces acting at a point



Let the two forces P and Q acting at A be represented by AB and AD. Let α be the angle between them.

i.e. $\angle BAD = \alpha$

Complete the parallelogram ABCD.

Then the diagonal AC will represent the resultant.

Let $\angle$CAB $= \varphi$

Draw CE $\perp r$ to AB. Now BC = AD = Q.

From the right angled ΔCBE,

$\sin C\hat{B}E = \frac{CE}{BC}$ i.e. $\sin\alpha = \frac{CE}{Q}$

$\therefore$ CE = Q $\sin\alpha$ .... (i)

$\cos\alpha = \frac{BE}{BC} = \frac{BE}{Q}$

$\therefore$ BE = Q $\cos\alpha$ .... (ii)

$R^2 = AC^2 = AE^2 + CE^2 = (AB + BE)^2 + CE^2$

$= (P + Q\cos\alpha)^2 + (Q\sin\alpha)^2$

$= P^2 + 2PQ\cos\alpha + Q^2$

$$\boxed{\therefore R = \sqrt[2]{\left(P^2 + 2PQ\cos\alpha + Q^2\right)}}$$

$$\boxed{\tan\varphi = \frac{CE}{AE} = \frac{Q\sin\alpha}{P + Q\cos\alpha}}$$

Result 1 If the forces P and Q are at right angles to each other, then $\alpha = 90^o$;

$R = \sqrt{P^2 + Q^2}$ $\quad \tan\varphi = \frac{Q}{P}$

Result 2 If the forces are equal (i.e.) Q = P, then

$R = \sqrt{P^2 + 2P^2\cos\alpha + P^2} = \sqrt{2P^2(1+\cos\alpha)}$

$= \sqrt{2P^2.2\cos^2\frac{\alpha}{2}} = 2P\cos\frac{\alpha}{2}$

$\tan\varphi = \frac{P\sin\alpha}{P + P\cos\alpha} = \frac{\sin\alpha}{1+\cos\alpha} = \frac{2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}}{2\cos^2\frac{\alpha}{2}}$

$= \tan\frac{\alpha}{2}$

ie) $\boxed{\varphi = \frac{\alpha}{2}}$

Thus the resultant of two equal forces P, P at an angle α is 2 P cos $\frac{\alpha}{2}$ in a direction bisecting the angle between them.

Result 3 Resultant R is greatest when cos α is greatest.

i.e. when cos α = 1 or $\alpha = 0^{o}$.

ie) Greatest value of R is R = P +Q.

R is least when cos α is least.

i.e. when cos $\alpha = -1$ or $\alpha = 180^{o}$. Least value of R is P~Q.

Problem 1

The resultant of two forces P, Q acting at a certain angle is X and that of P, R acting at the same angle is also X. The resultant of Q, R again acting at the same angle is Y, Prove that.

$$P = (X^2 + QR)^{1/2} = \frac{QR(Q+R)}{Q^2 + R^2 - Y^2}$$

Prove also that, if P + Q + R = 0, Y = X.

Solution:

Let α be the angle between P and Q

Given

$X^2 = P^2 + Q^2 + 2PQ\cos\alpha$ .... (1)

$X^2 = P^2 + R^2 + 2PR\cos\alpha$ .... (2)

$Y^2 = Q^2 + R^2 + 2QR\cos\alpha$ .... (3)

(1) – (2) gives $0 = Q^2 - R^2 + 2P\cos\alpha\,(Q-R)$

i.e. $0 = (Q-R)(Q+R+2P\cos\alpha)$

But $Q \neq R$ and so $Q - R \neq 0$

$\therefore Q + R + 2P\cos\alpha = 0$

$$\cos\alpha = -\frac{Q+R}{2P} \qquad \text{........ (4)}$$

Substitute (4) in (1),

$$X^2 = P^2 + Q^2 + 2PQ\left[-\left(\frac{Q+R}{2P}\right)\right] = P^2 + Q^2 - Q^2 - QR$$

$P^2 = X^2 + QR$. i.e. $P = (X^2 + QR)^{1/2}$

Substitute (4) in (3),

$$Y^2 = Q^2 + R^2 + 2QR\left[-\left(\frac{Q+R}{2P}\right)\right]$$

$$= Q^2 + R^2 - \frac{QR(Q+R)}{P}$$

$$\therefore \frac{QR(Q+R)}{P} = Q^2 + R^2 - Y^2$$

$$P = \frac{QR(Q+R)}{Q^2 + R^2 - Y^2}$$

If P + Q + R = 0, then Q + R = $-P$,

$$\therefore \text{From (4)}, \cos\alpha = -\frac{Q+R}{2P} = \frac{P}{2P} = \frac{1}{2}$$

$$\cos\alpha = \frac{1}{2} \Rightarrow$$

$$X^2 = P^2 + R^2 + PR \ldots \quad \ldots \quad \ldots (5)$$

$$Y^2 = Q^2 + R^2 + QR \ldots \quad \ldots \quad \ldots (6)$$

(5) – (6) gives

$$X^2 - Y^2 = P^2 - Q^2 + PR - QR$$

$$= (P - Q)(P + Q + R)$$

$$= (P - Q).0 = 0$$

$$\therefore X = Y$$

Problem 2

Two forces of given magnitude P and Q act at a point at an angle α. What will be the maximum and minimum value of the resultant?

Solution:

i. Maximum value of the resultant = P + Q

ii. Minimum value of the resultant = P~ Q.

Problem 3

The greatest and least magnitudes of the resultant of two forces of constant magnitudes are R and S respectively. Prove that, when the forces act at an angle 2φ, the resultant is of magnitude $\sqrt{R^2\cos^2\varphi + S^2\sin^2\varphi}$

Solution:

Given, R = P + Q, S = P-Q, where P and Q are two forces.

When P and Q are acting at an angle 2φ

$$\text{Resultant} = \sqrt{P^2 + Q^2 + 2PQ.\cos 2\varphi}$$

$$= \sqrt{\left(P^2 + Q^2\right) + 2PQ\left(\cos^2\varphi - \sin^2\varphi\right)}$$

$$= \sqrt{\left(P^2 + Q^2\right)\left(\sin^2\varphi + \cos^2\varphi\right) + 2PQ\left(\cos^2\varphi - \sin^2\varphi\right)}$$

$$= \sqrt{\left(P^2 + Q^2 + 2PQ\right)\cos^2\varphi + \left(P^2 + Q^2 - 2PQ\right)\sin^2\varphi}$$

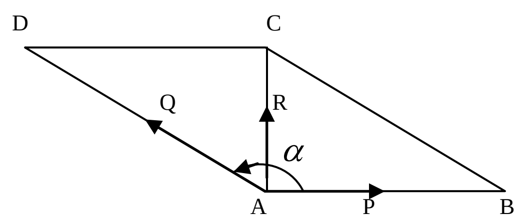
$$= \sqrt{R^2\cos^2\varphi + S^2\sin^2\varphi}.$$

Problem 4

The resultant of two forces P and Q is at right angles to P. Show that the angle between the forces is $\cos^{-1}\left(-\frac{P}{Q}\right)$

Solution:

Let α be the angle between the two forces P and Q. Given φ = $90^{o.}$



We know, $\tan\varphi = \dfrac{Q\sin\alpha}{P + Q\cos\alpha}$

i.e. $\tan 90^o = \dfrac{Q\sin\alpha}{P + Q\cos\alpha}$

$$\frac{1}{0} = \frac{Q\sin\alpha}{P+Q\cos\alpha}$$

$$\therefore P+Q\cos\alpha = 0$$

$$\therefore \cos\alpha = -\frac{P}{Q}$$

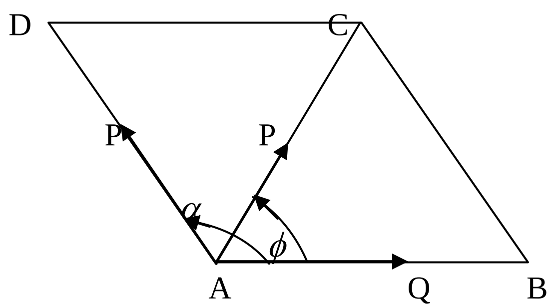
$$\boxed{\therefore \alpha = \cos^{-1}\left(-\frac{P}{Q}\right)}$$

Problem 5

The resultant of two forces P and Q is of magnitude P. Show that, if P be doubled, the new resultant is at right angles to Q and its magnitude will be $\sqrt{4P^2 - Q^2}$.

Solution:

Let α be the angle between P and Q



Given, $P^2 = P^2 + Q^2 + 2PQ\cos\alpha$.

$\therefore$ Q (Q+2Pcosα) = 0

$$\boxed{\therefore \cos\alpha = -\frac{Q}{2P}}$$

If **P is doubled,** let R be the new resultant, and φ be the angle between Q and R.

$$\therefore R^2 = (2P)^2 + Q^2 + 2(2P)Q.\cos\alpha$$

$$= 4P^2 + Q^2 + 4PQ\left(-\frac{Q}{2P}\right)$$

$$= 4P^2 + Q^2 - 2Q^2 = 4P^2 - Q^2$$

$$\boxed{\therefore R = \sqrt{4P^2 - Q^2}}$$

$$\tan\varphi = \frac{(2P)\sin\alpha}{Q+(2P)\cos\alpha} = \frac{2P\sin\alpha}{Q+2P\left(-\frac{Q}{2P}\right)}$$

i.e. $\tan\varphi = \frac{2P\sin\alpha}{0}$

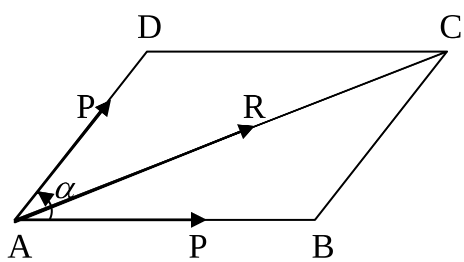
$\therefore \cos\varphi = 0 \Rightarrow \varphi = 90^0$

$\therefore$ Q is at right angles to R.

Problem 6

Two equal forces act on a particle, find the angle between them when the square of their resultant is equal to three times their product.

Solution:



Let α be the angle between the two equal forces P, P, and let R be their resultant.

$$\therefore R^2 = P^2 + P^2 + 2P.P.\cos\alpha$$

$$= 2P^2(1+\cos\alpha) = 2P^2 \times 2\cos^2\frac{\alpha}{2}$$

i.e. $R^2 = 4P^2\cos^2\frac{\alpha}{2}$

$$\therefore \boxed{R = 2P\cos\frac{\alpha}{2}}$$

Given, $R^2 = 3\times P\times P = 3P^2$

$$\therefore 3P^2 = 4P^2\cos^2\frac{\alpha}{2}$$

$$\therefore \cos^2\frac{\alpha}{2} = \frac{3}{4} \Rightarrow \cos\frac{\alpha}{2} = \frac{\sqrt{3}}{2}$$

$$\Rightarrow \quad \frac{\alpha}{2} = 30^{o}$$

$$\Rightarrow \alpha = 60^{0}$$

Problem 7

If the resultant of forces 3P, 5P is equal to 7P find

i. the angle between the forces

ii. the angle which the resultant makes with the first force.

Solution:

Let α be the angle between 3P, 5P

i. Given $(7P)^2 = (3P)^2 + (5P)^2 + 2\,(3P)\,(5P)\,.\cos\alpha$

$$49P^2 = 9P^2 + 25P^2 + 30P^2\cos\alpha$$

$$\therefore 15P^2 = 30P^2\cos\alpha$$

$$\therefore \cos\alpha = \frac{1}{2} \quad \Rightarrow \alpha = 60^{0}$$

ii. Let φ be the angle between the resultant and 3P.

$$\therefore \tan\varphi = \frac{Q\sin\alpha}{P + Q\cos\alpha}$$

$$= \frac{5P.\sin\alpha}{3P + 5P.\cos\alpha}$$

$$= \frac{5P.\sin 60^\circ}{3P + 5P.\cos 60^\circ}$$

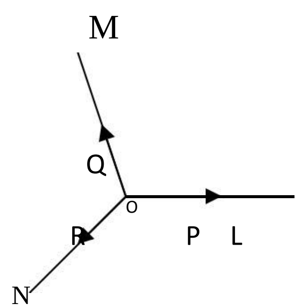
$$= \frac{5 \times \frac{\sqrt{3}}{2}}{3 + \left(5 \times \frac{1}{2}\right)}$$

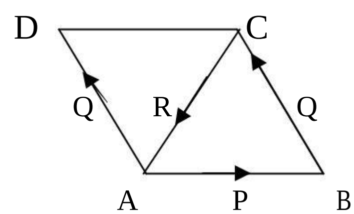
$$\tan\varphi = \frac{5\sqrt{3}}{11}$$

$$\therefore \varphi = \tan^{-1}\left(\frac{5\sqrt{3}}{11}\right)$$

1.2 Triangle of forces

If three forces acting at a point can be represented in magnitude and direction by the sides of a triangle taken in order, they will be in equilibrium.





Let the forces, P,Q,R act at a point O and be represented in magnitude and direction by the sides AB,BC,CA of the triangle ABC.

To prove : They will be in equilibrium.

Complete the parallelogram BADC.

P+Q $= \overline{AB} + \overline{AD} = \overline{AB} + \overline{BC}$

$= \overline{AC}$

ie) The resultant of the forces P, Q at O is represented in magnitude and direction by AC.

The third force R acts at O and it is represented in magnitude and direction by CA.

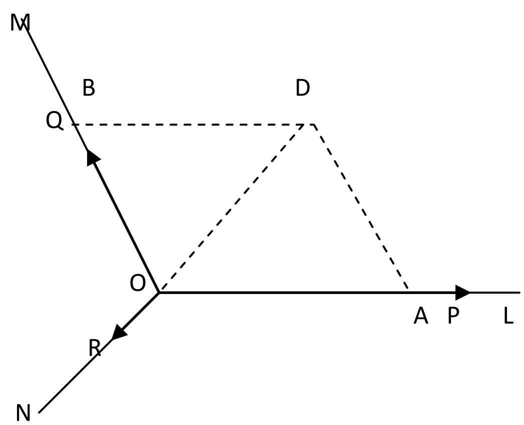
Hence P+Q+R = $\overline{AC} + \overline{CA} = \overline{0}$

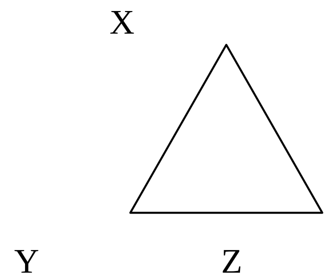
Principle

If two forces acting at a point are represented in magnitude and direction by two sides of a triangle taken in the same order, the resultant will be represented in magnitude and direction by the third side taken in the reverse order.

1.3 Lami's Theorem

If three forces acting at a point are in equilibrium, each force is proportional to the sine of the angle between the other two.





Proof:

By converse of the triangle of forces, the sides of the triangle OAD represent the forces P,Q,R in magnitude and direction.

By sine rule in ΔOAD, we have

$$\frac{OA}{\sin\angle ODA} = \frac{AD}{\sin\angle DOA} = \frac{DO}{\sin\angle OAD} \quad \ldots\ldots\ldots\ldots\ldots (1)$$

But $\angle OAD = alt.\angle BOD = 180^0 - \angle MON$

$$\therefore \sin\angle ODA = \sin\left(180^0 - \angle MON\right) = \sin\angle MON \quad \ldots\ldots (2)$$

Also $\angle DOA = 180^0 - \angle NOL$

$$\therefore \sin\angle DOA = \sin\left(180^0 - \angle NOL\right) = \sin\angle NOL \quad \ldots\ldots (3)$$

And $\angle OAD = 180^0 - \angle BOA = 180^0 - \angle LOM$

$\therefore \sin \angle OAD = \sin\left(180^0 - \angle LOM\right) = \sin \angle LOM$ ……. (4)

Substitute (2), (3), (4) in (1),

$$\frac{OA}{\sin \angle MON} = \frac{AD}{\sin \angle NOL} = \frac{DO}{\sin \angle LOM}$$

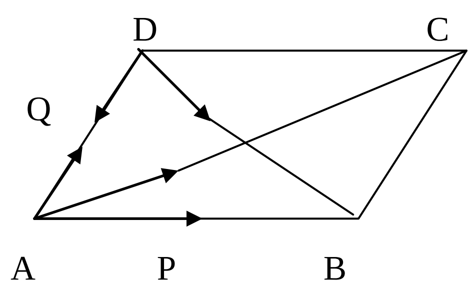
i.e. $\frac{P}{\sin \angle MON} = \frac{Q}{\sin \angle NOL} = \frac{R}{\sin \angle LOM}$

$$\frac{P}{\sin(Q.R)} = \frac{Q}{\sin(R,P)} = \frac{R}{\sin(P,Q)}$$

Problem 8

Two forces act on a particle. If the sum and difference of the forces are at right angles to each other, show that the forces are of equal magnitude.

Solution:



Let the forces P and Q acting at A be represented in magnitude and direction by the lines AB and AD. Complete the parallelogram BAD.

Then P+Q= $\overline{AB} + \overline{AD} = \overline{AC}$

P-Q $= \overline{AB} - \overline{AD}$

$= \overline{AB} + \overline{DA}$

$= \overline{DA} + \overline{AB}$

$= \overline{DB}$

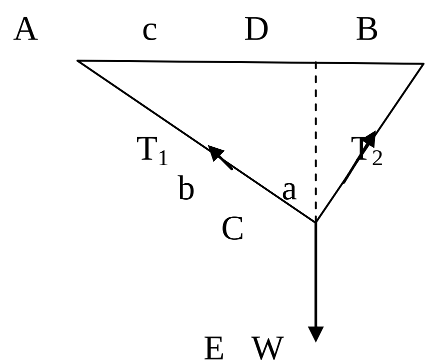
Given $\overline{AC}$ and $\overline{DB}$ are at right angles.
The diagonals AC and BD cut at right angles.

$\therefore$ ABCD must be a rhombus.
$\therefore$ AB = AD.
P = Q.

Problem 9

Let A and B two fixed points on a horizontal line at a distance c apart. Two fine light strings AC and BC of lengths b and a respectively support a mass at C. Show that the tensions of the strings are in the ratio $b\left(a^2+c^2-b^2\right):a\left(b^2+c^2-a^2\right)$

Solution



Forces T_1, T_2, W are acting at C.

By Lami's theorem,

$$\frac{T_1}{\sin\angle ECB}=\frac{T_2}{\sin\angle ECA}\text{........(1)}$$

Now $\sin\ \angle ECB=\sin\left(180^0-\angle DCB\right)$

$= \sin\ \angle DCB$

$= \sin\ \left(90^0-\angle ABC\right)=\cos\angle ABC$

$\sin\ \angle ECA=\sin\left(180^0-\angle ACD\right)$

$= \sin\ \angle ACD$

$= \sin\ \left(90^0-\angle BAC\right)=\cos\angle BAC$

$$\frac{T_1}{\cos\angle ABC}=\frac{T_2}{\cos\angle BAC}\quad\therefore\quad\frac{T_1}{T_2}=\frac{\cos B}{\cos A}=\frac{\left(\dfrac{c^2+a^2-b^2}{2ca}\right)}{\left(\dfrac{b^2+c^2-a^2}{2bc}\right)}$$

$$\therefore\frac{T_1}{T_2}=\left(\frac{c^2+a^2-b^2}{2ca}\right)\times\left(\frac{2bc}{b^2+c^2-a^2}\right)=\frac{b\left(c^2+a^2-b^2\right)}{a\left(b^2+c^2-a^2\right)}$$

Problem 10

ABC is a given triangle. Forces P,Q,R acting along the lines OA,OB,OC are in equilibrium. Prove that

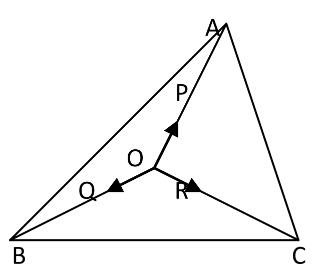
(i)P : Q : R= $a^2\left(b^2+c^2-a^2\right):b^2\left(c^2+a^2-b^2\right):c^2\left(a^2+b^2-c^2\right)$ if O is the cicumcentre of the triangle.

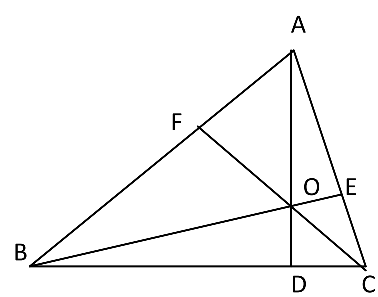
(ii) P : Q : R= $=\cos\frac{A}{2}:\cos\frac{B}{2}:\cos\frac{C}{2}$ if O is the incentre of the triangle.

(iii) P : Q : R= a:b:c if O is the ortho centre of the triangle.

(iv) P : Q : R=OA : OB : OC if O is the centroid of the triangle,

Solution:





By Lami's theorem,

$$\frac{P}{\sin\angle BOC}=\frac{Q}{\sin\angle COA}=\frac{R}{\sin\angle AOB}\qquad\text{……………} (1)$$

(i) O is the circumcentre of the Δ ABC

$$\angle BOC=2\angle BAC=2A;\angle COA=2B\ \ and\ \ \angle AOB=2C$$

$$\therefore (1) \Rightarrow \quad \frac{P}{\sin 2A} = \frac{Q}{\sin 2B} = \frac{R}{\sin 2C}$$

i.e. $$\frac{P}{2\sin A\cos A} = \frac{Q}{2\sin B\cos B} = \frac{R}{2\sin C\cos C} \quad \ldots\ldots (2)$$

But $\cos A = \dfrac{b^2+c^2-a^2}{2bc}$ and $\sin A = \dfrac{2\Delta}{bc}$

where Δ is the area of the triangle ABC

$$\therefore 2\sin A\cos A = 2\,\frac{2\Delta\left(b^2+c^2-a^2\right)}{bc \quad 2bc}$$

$$= \frac{2\Delta\left(b^2+c^2-a^2\right)}{b^2c^2}$$

Similarly $2\sin B\cos B = \dfrac{2\Delta\left(c^2+a^2-b^2\right)}{c^2a^2}$

$$2\sin C\cos C = \frac{2\Delta\left(a^2+b^2-c^2\right)}{a^2b^2}$$

Substitute in (2)

$$\frac{P.b^2c^2}{2\Delta\left(b^2+c^2-a^2\right)} = \frac{Q.c^2a^2}{2\Delta\left(c^2+a^2-b^2\right)} = \frac{Ra^2b^2}{2\Delta\left(a^2+b^2-c^2\right)}$$

Divide by $\dfrac{a^2b^2c^2}{2\Delta}$

$$\frac{P}{a^2\left(b^2+c^2-a^2\right)} = \frac{Q}{b^2\left(c^2+a^2-b^2\right)} = \frac{R}{c^2\left(a^2+b^2-c^2\right)}$$

(ii) O is the in-centre of the triangle,

OB and OC are the bisectors of $\angle$B and $\angle$C

$$\therefore \angle BOC = 180^0 - \frac{B}{2} - \frac{C}{2} = 180^0 - \left(\frac{B}{2} + \frac{C}{2}\right)$$

$$= 180^0 - \left(90^0 - \frac{A}{2}\right) = 90^0 + \frac{A}{2}$$

Similarly $\angle$COA = $90^0 + \frac{B}{2}$, $\angle AOB = 90^0 + \frac{C}{2}$

(1) $\Rightarrow \dfrac{P}{\sin\left(90^0 + \frac{A}{2}\right)} = \dfrac{Q}{\sin\left(90^0 + \frac{B}{2}\right)} = \dfrac{R}{\sin\left(90^0 + \frac{C}{2}\right)}$

i.e. $\dfrac{P}{\cos\frac{A}{2}} = \dfrac{Q}{\cos\frac{B}{2}} = \dfrac{R}{\cos\frac{C}{2}}$

(iii) O is the ortho-centre of the triangle

AD, BE, CF are the altitudes of the triangle

AFOE is a cyclic quadrilateral.

$\therefore \angle FOE + A = 180^0$, $\therefore \angle FOE = 180^0 - A$

$\therefore \angle BOC = 180^0 - A$

Similarly, $\angle COA = 180^0 - B$, $\angle AOB = 180^0 - C$

Hence (1) becomes

$$\frac{P}{\sin\left(180^0 - A\right)} = \frac{Q}{\sin\left(180^0 - B\right)} = \frac{R}{\sin\left(180^0 - C\right)}$$

i.e. $\dfrac{P}{\sin A} = \dfrac{Q}{\sin B} = \dfrac{R}{\sin C}$

i.e. $\dfrac{P}{a} = \dfrac{Q}{b} = \dfrac{R}{c}$ $\left(\because \dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}\right)$

(iv) O is the centroid of the triangle

Δ BOC = Δ COA = Δ AOB = $\frac{1}{3}\Delta ABC$

Δ BOC = $\frac{1}{2}OB.OC\sin\angle BOC = \frac{1}{3}\Delta ABC$

$\therefore \sin\angle BOC = \frac{2\Delta ABC}{3OB.OC}$

Similarly, $\sin\angle COA = \frac{2\Delta ABC}{3OC.OA}$, $\sin\angle AOB = \frac{2\Delta ABC}{3OA.OB}$

Hence (1) becomes $\frac{P.3OB.OC}{2\Delta ABC} = \frac{Q.3OC.OA}{2\Delta ABC} = \frac{R.3OA.OB}{2\Delta ABC}$

i.e. P.OB.OC = Q.OC.OA = R.OA.OB

Dividing by OA.OB.OC, we get $\frac{P}{OA} = \frac{Q}{OB} = \frac{R}{OC}$.

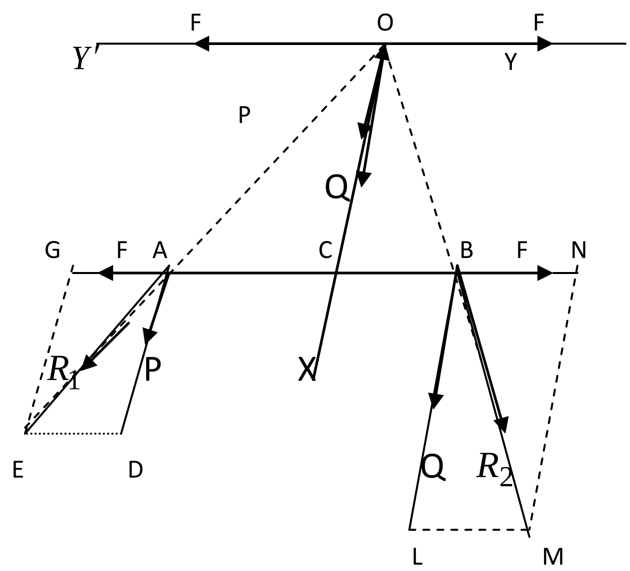
1.4 Parallel forces:

Forces acting along parallel lines are called parallel forces. There are two types of parallel forces known as like and unlike parallel forces. Since the parallel forces do not meet at a point, in this chapter we study methods to find the resultant of two like parallel and unlike parallel forces. Parallel forces acting on a rigid body have a tendency to rotate it about a fixed point. Such tendency is known as moment of the parallel forces. Here we study the theorem on moments of forces about a point.

Definition:

Two parallel forces are said to be **like** if they act in the same direction, they are said to be **unlike** if they act in opposite parallel directions.

The resultant of two like parallel forces acting on a rigid body



Proof:

Let P and Q be two like parallel forces acting at A and B along the lines AD and BL.At A and B, introduce two equal and opposite forces F along AG and BN. These two forces F balance each other and will not affect the system.

Now, R_1 is the resultant of P and F at A and R_2 is the resultant of Q and F at B as in the diagram.

Produce EA and MB to meet at O. At O, draw YOY^1 parallel to AB and draw OX parallel to the direction of P.

Resolve R_1 and R_2 at O into their original components. R_1 at O is equal to F along OY^1 and P along OX. R_2 at O is equal to F along OY and Q along OX.

The two forces F, F at O cancel each other. The remaining two forces P and Q acting along OX have the resultant **P+Q (sum)** along OX.

Find the position of the resultant

Now, AB and OX meet at C.

Triangles, OAC and AED are similar.

$$\therefore \ \frac{OC}{AD} = \frac{AC}{ED} \text{ ie) } \frac{OC}{P} = \frac{AC}{F}$$

$$\therefore \ F.OC \ = P.AC \ \ldots\ldots\ldots\ldots\ldots\ldots \ (1)$$

Triangles OCB and BLM are similar.

$$\therefore \ \frac{OC}{BL} = \frac{CB}{LM} \text{ ie) } \frac{OC}{Q} = \frac{CB}{F}$$

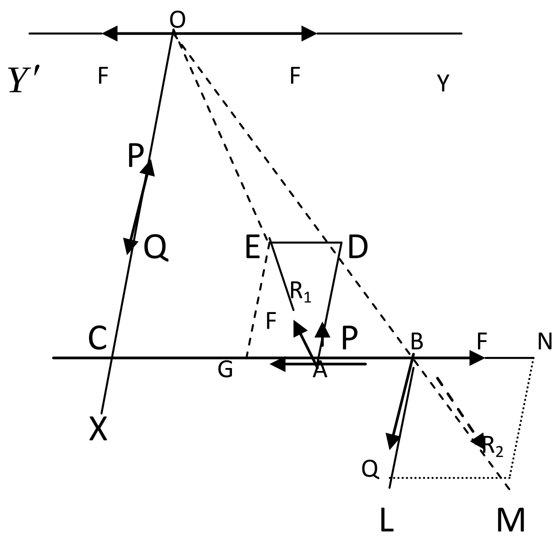
$$\therefore \ F.OC = Q.CB \ \ldots\ldots\ldots\ldots\ldots\ldots \ (2)$$

(1) & (2) $\Rightarrow$ $\boxed{\text{P.AC = Q.CB}}$

ie) $\dfrac{AC}{CB} = \dfrac{Q}{P}$

ie) ‘C’ divides AB internally in the inverse ratio of the forces.

The resultant of two unlike and unequal parallel forces acting on a rigid body:



Proof:

Let P and Q at A and B be two unequal unlike parallel forces acting along AD and BL.

Let P > Q.

At A and B introduce two equal and opposite forces F along AG and BN. These two balances each other and will not affect the system.

Let R_1 be the resultant of F and P at A and R_2 be the resultant of F and Q at B. as in the diagram.

Produce EA and MB to meet at O. At O, draw Y'OY parallel to AB and draw OX parallel to the direction of P.

Resolve R_1 and R_2 at O into their components. R_1 at O is equal to F along OY' and P along XO. R_2 at O is equal to F along OY and Q along OX.

The two forces F, F at O cancel each other. Now, the remaining forces are P and Q along the same line but opposite directions.

Hence the resultant is **P ~ Q (difference)** along XO.

Find the position of the resultant

Now, AB and OX meet at C.

Triangles OCA and EGA are similar.

$$\therefore \frac{OC}{EG} = \frac{CA}{GA}, \text{ ie) } \frac{OC}{P} = \frac{CA}{F}$$

$$F.OC = P.AC \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

Triangles OCB and BLM are similar.

$$\therefore \frac{OC}{BL} = \frac{CB}{LM}, \text{ ie) } \frac{OC}{Q} = \frac{CB}{F}$$

$$\therefore F.OC = Q.CB \quad \ldots\ldots\ldots\ldots\ldots\ldots (2)$$

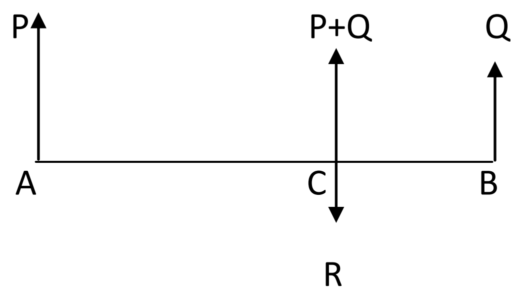
(1) and (2) $\Rightarrow$ $\boxed{\text{P.AC = Q.CB}}$

ie) $\frac{CA}{CB} = \frac{Q}{P}$

ie) ‘C’ divides AB externally.

Note : The effect of two equal and unlike parallel forces can not be replaced by a single force.

The condition of equilibrium of three coplanar parallel forces



Let P, Q, R be the three coplanar parallel forces in equilibrium. Draw a line to meet the forces P, Q, R at the points A, B, C respectively.

Equilibrium is not possible if all the three forces are in the same direction.

Let P + Q be the resultant of P and Q parallel to P. Hence R must be equal and opposite to P + Q.

$\therefore$ R = P + Q (in magnitude, opposite in direction)

$\therefore P.AC = Q.CB$

$$\therefore \frac{P}{CB} = \frac{Q}{AC} = \frac{P+Q}{CB+AC} = \frac{R}{AB}$$

Hence, $\boxed{\dfrac{P}{CB} = \dfrac{Q}{AC} = \dfrac{R}{AB}}$

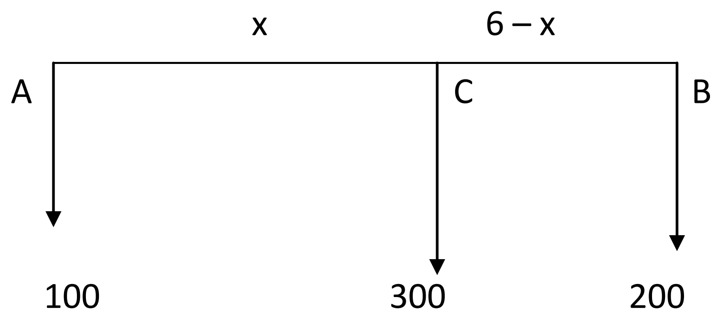
ie) If three parallel forces are in equilibrium then each force is proportional to the distance between the other two.

Note: The centre of two parallel forces is a fixed point through which their resultant always passes.

Problem 11

Two men, one stronger than the other, have to remove a block of stone weighing 300 kgs. with a light pole whose length is 6 metre. The weaker man cannot carry more than 100 kgs. Where the stone be fastened to the pole, so as just to allow him his full share of weight?

Solution:



Let A be the weaker man bearing 100 kgs., B the stronger man bearing 200 kgs. Let C be the point on AB where the stone is fastened to the pole, such that AC = x. Then the weight of the stone acting at C is the resultant of the parallel forces 100 and 200 at A and B respectively.

$\therefore$ 100.AC = 200.BC

i.e. 100x = 200 (6-x) = 1200 – 200x

$\therefore$ 300x = 1200 or x=4

Hence the stone must be fastened to the pole at the point distant 4 metres from the weaker man.

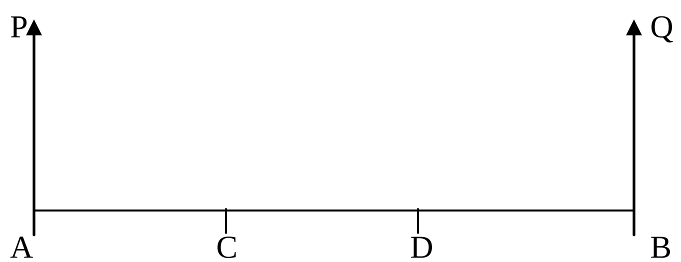
Problem 12

Two like parallel forces P and Q act on a rigid body at A and B respectively.

a) If Q be changed to $\frac{P^2}{Q}$, show that the line of action of the resultant is the same as it would be if the forces were simply interchanged.

b) If P and Q be interchanged in position, show that the point of application of the resultant will be displayed along AB through a distance d, where $d = \frac{P-Q}{P+Q}.AB$.

Solution:



Let C – be the centre of the two forces.

Then P. AC = Q.CB …………. (1)

(a) If Q is changed to $\frac{P^2}{Q}$, (P remaining the same), let D be the new centre of parallel forces.

Then P.AD = $\frac{P^2}{Q}$ DB …. …. ….. (2)

Q.AD = P.DB ……………. (3)

Relation (3) shows that D is the centre of two like parallel forces, with Q at A and P at B.

(b) When the forces P and Q are interchanged in position, D is the new centre of parallel forces.

Let CD = d

From (3), Q. (AC+CD) = P. (CB – CD)

i.e. Q.AC + Q.d = P.CB – P.d

(Q + P).d = P.CB – Q.AC

= P (AB – AC) – Q (AB – CB)

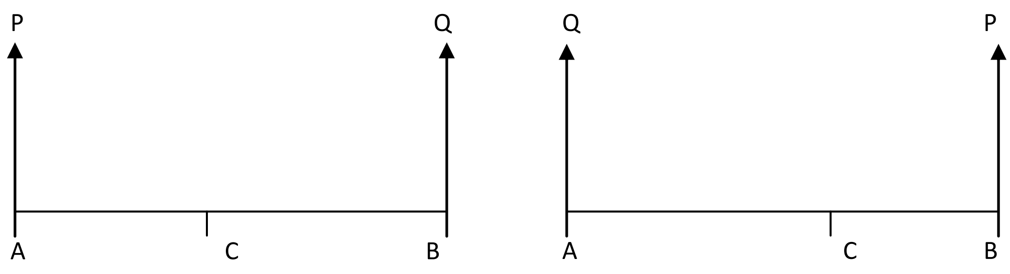
= (P – Q).AB[$\because$ P.AC = Q.CB from (1)]

$$\boxed{d = \frac{P-Q}{P+Q}.AB}$$

Problem 13

The position of the resultant of two like parallel forces P and Q is unaltered, when the position of P and Q are interchanged. Show that P and Q are of equal magnitude.

Solution:



Let C be the centre of two like parallel forces P at A and Q at B.

$\therefore$ P.AC = Q.CB ……………… (1)

When P and Q are interchanged, the centre C is not altered (given)

$\therefore$ Q.AC =P.CB ………………. (2)

$$\frac{(1)}{(2)} \Rightarrow \frac{P}{Q} = \frac{Q}{P}$$

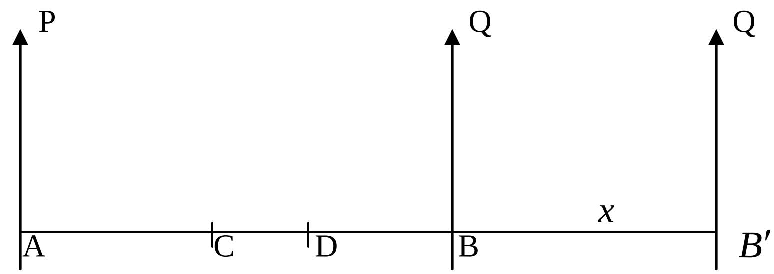
$$\therefore P^2 = Q^2$$

$$\boxed{\therefore P = \pm \ Q}$$

Problem 14

P and Q are like parallel forces. If Q is moved parallel to itself through a distance x, prove that the resultant of P and Q moves through a distance $\frac{Qx}{P+Q}$.

Solution:



Let C be the centre of P and Q at A and B.

$\therefore P.AC = Q.CB$ ………… (1)

Let D be the new centre of P at A and Q at B' such that BB' = x

$\therefore P.AD = Q.DB'$ …………………… (2)

ie) $P(AC + CD) = Q[DB + BB']$ = $Q[(CB - CD) + x]$

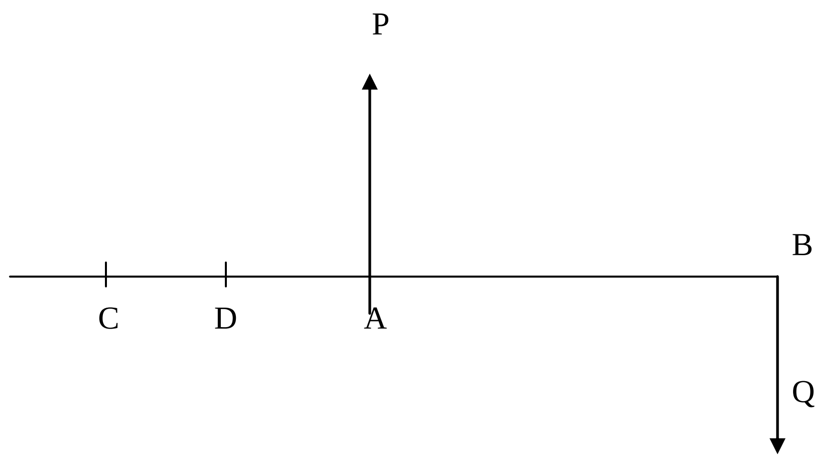
$\therefore (P+Q)CD = Q.x$ using (1)

$$\boxed{\therefore CD = \frac{Qx}{P+Q}}$$

Problem 15

Two unlike parallel forces P and Q (P>Q) acting on a rigid body at A and B respectively be interchanged in position, show that the point application of the resultant in AB will be displayed along AB through a distance $\frac{P+Q}{P-Q}AB$.

Solution:



Let C be the centre of two unlike parallel forces P at A and Q at B.

$\therefore P.AC = Q.CB$ ………………… (1)

Let D be the new centre when P and Q are interchanged in position.

$\therefore Q.AD = P.DB$ ……………….. (2)

i.e.) $Q(AC - CD) = P.(DA + AB)$

i.e.) $Q[(CB - AB) - CD] = P.[(AC - CD) + AB]$

$Q.CB - Q.AB - Q.CD = P.AC - P.CD + P.AB$

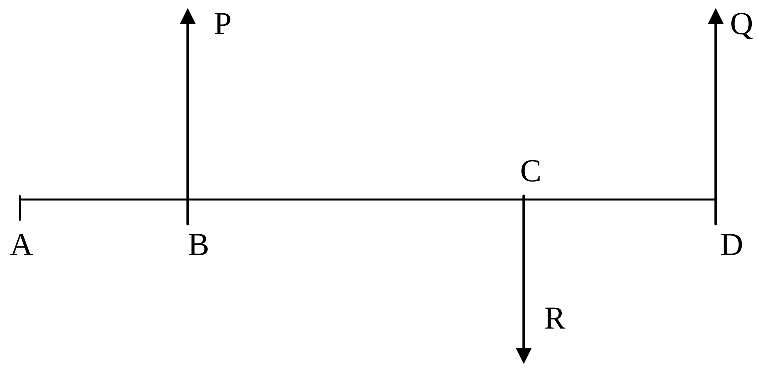
$\therefore (P-Q).CD = (P+Q).AB$ using (1)

$$\boxed{\therefore CD = \frac{P+Q}{P-Q}.AB}$$

Problem 16

A light rod is acted on by three parallel forces P, Q, and R, acting at three points distant 2, 8 and 6 ft. respectively from one end. If the rod is in equilibrium, show that P: Q: R = 1:2:3.

Solution



P, Q, R are parallel forces acting on the rod AD at B, D, C respectively.

Given, AB = 2 ft, AD = 8ft, AC = 6ft.

$\therefore$ BC = 4ft, CD = 2ft, BD = 6ft.

For equilibrium of the rod, each force should be proportional to the distance between the other two.

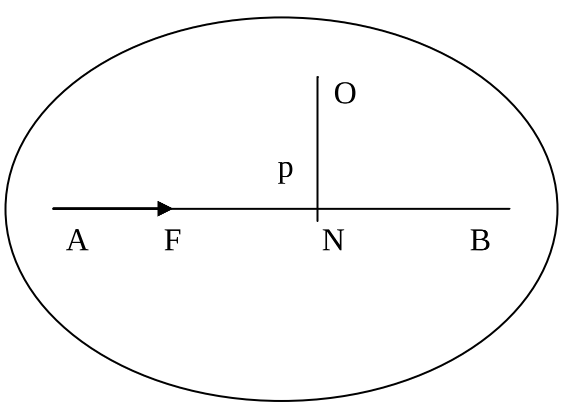
$$\therefore \frac{P}{2} = \frac{Q}{4} = \frac{R}{6} \Rightarrow P:Q:R = 2:4:6$$

$$\boxed{\therefore P:Q:R = 1:2:3}$$

1.5 Moment of a force (or) Turning effect of a force

Definition:

The moment of a force about a point is defined as the product of the force and the perpendicular distance of the point from the line of action of the force.



Moment of F about O = F x ON = F x p.

Note: Moment of F about O is zero if either F = O (or) ON = O.

i.e.) F = 0 (or) AB passes through O.

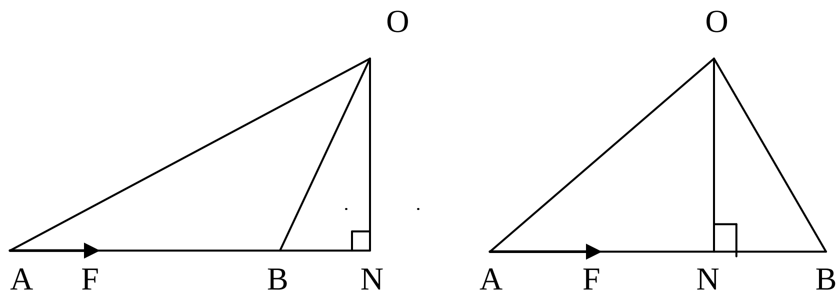
Hence, moment of a force about any point is zero if either

the force itself is zero (or) the force passes through that point.

Physical significance of the moment of a force

It measures the tendency to rotate the body about the fixed point.

Geometrical Representation of a moment



Let AB represent the force F both in magnitude and direction and O be any given point.

$\therefore$ the moment of the force F about O

= F x ON = AB x ON = 2. Δ AOB

= Twice the area of the triangle AOB

Sign of the moment

If the force tends to turn the body in the **anticlockwise** direction, moment is **positive.**

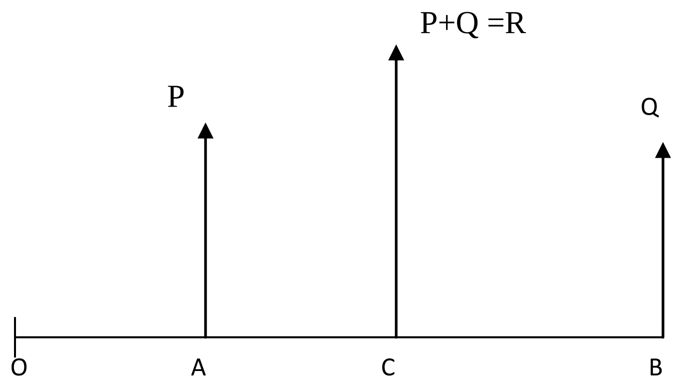
If the force tends to turn the body in the **clockwise** direction, moment is **negative.**

Varignon's Theorem of Moments

The algebraic sum of the moments of two forces about any point in their plane is equal to the moment of their resultant about that point.

Proof:

Case 1 Let the forces be parallel and O lies **i) Outside AB**



Let P and Q be the two parallel forces acting at A and B. P + Q be their resultant R acting at C. such that

P.AC = Q.CB ……………... (1)

Algebraic sum of the moments of P and Q about O

= P.OA + Q.OB

= P x (OC – AC) + Q x (OC + CB)

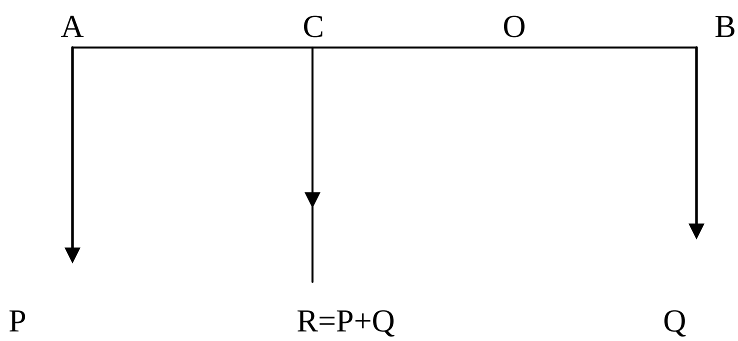
= (P +Q).OC – P.AC +Q.CB

= (P+Q).OC using (1)

= R.OC

= moment of R about O.

ii) P and Q are parallel and O lies within AB



Algebraic sum of the moments of P and Q about O

= P.OA – Q.OB

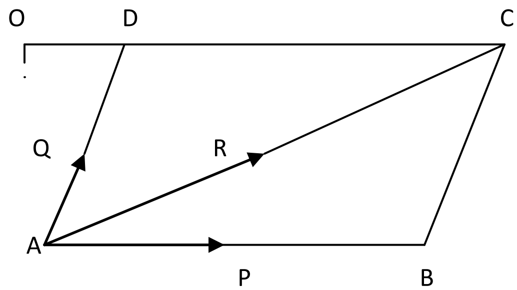
= P. (OC+CA) – Q. (CB – CO)

= (P+Q).OC + P.CA – Q.CB by (1)

= R.OC

= moment of R about O.

Case II iii) P and Q meet at a point and O any point in their plane. O lies outside the angle BAD



Through O, draw a line parallel to the direction of P, to meet the line of action of Q at D. Complete the parallelogram ABCD such that AB, AD represent the magnitude of P and Q and the diagonal AC represents the resultant R of P and Q.

Algebraic sum of the moments of P and Q about O

$= 2.\ \Delta\ AOB + 2.\Delta\ AOD$

$= 2\ \Delta\ ACB + 2.\ \Delta\ AOD\ [\because \Delta\ AOB = \Delta\ ACB]$

$= 2\ \Delta\ ADC + 2\ \Delta\ AOD$

$= 2\ (\Delta\ ADC + \Delta\ AOD)$

$= 2.\ \Delta\ AOC$

= Moment of R about O.

iv) O lies inside the angle BAD

Algebraic sum of the moments of P and Q about O:

$= 2\ \Delta\ AOB - 2\ \Delta\ AOD$

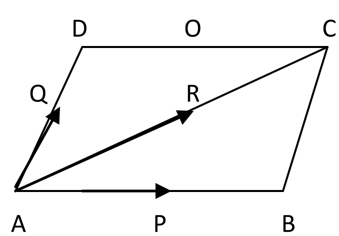
$= 2\ \Delta\ ACB - 2\ \Delta\ AOD$

$= 2\ \Delta\ ADC - 2\ \Delta\ AOD$

$= 2\ (\Delta\ ADC - \Delta\ AOD)$

$= 2.\ \Delta\ AOC$

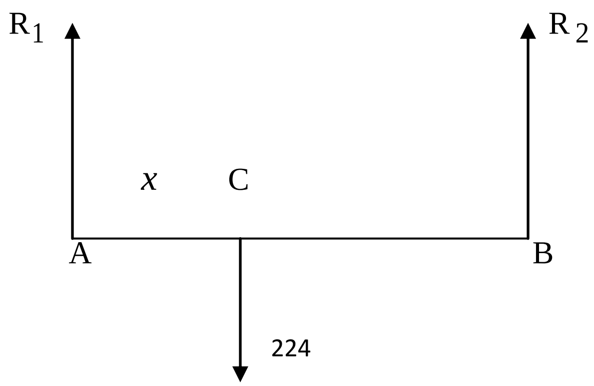
= moment of R about O.



Problem 17

Two men carry a load of 224 kg. wt, which hangs from a light pole of length 8 m. each end of which rests on a shoulder of one of the men. The point from which the load is hung is 2m. nearer to one man than the other. What is the pressure on each shoulder?

Solution



AB is the light pole of length 8m. C is the point from which the load of 224 kgs. is hung.

Let AC = x. Then BC = 8 – x. given (8 – x) – x =2

i.e) 8 – 2x = 2 0r 2x = 6.

$\therefore$ x = 3. i.e. AC = 3 and BC = 5.

Let the pressures at A and B be R_1 and R_2 kg. wt. respectively. Since the pole is in equilibrium, the algebraic sum of the moments of the three forces R_1, R_2 and 224 kg. wt. about any point must be equal to zero.

Taking moments about B,

224 CB – R_1.AB = 0

i.e. $224 \times 5 - R_1 \times 8 = 0.$

$$\therefore R_1 = \frac{224 \times 5}{8} = 140.$$

Taking moments about A,

R_2.AB – 224.AC = 0.

i.e. $8R_2 - 224 \times 3 = 0.$

$$\therefore R_2 = \frac{224 \times 3}{8} = 84$$

Problem 18

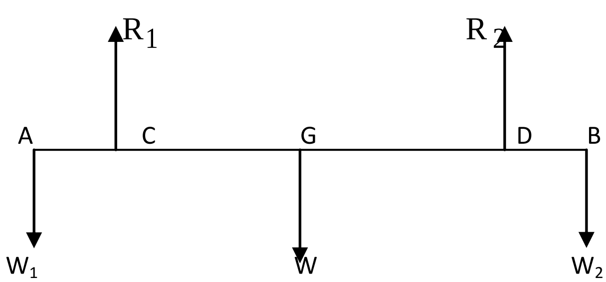
A uniform plank of length 2a and weight W is supported horizontally on two vertical props at a distance b apart. The greatest weight that can be placed at the two ends in succession without upsetting the plank are W_1 and W_2 respectively. Show that $\frac{W_1}{W+W_1}+\frac{W_2}{W+W_2}=\frac{b}{a}$.

Solution

Let AB be the plank placed upon two vertical props at C and D. CD = b. The weight W of the plank acts at G, the midpoint of AB,

AG = GB = a

When the weight W_1 is placed at A, the contact with D is just broken and the upward reaction at D is zero.



There is upward reaction R_1 at C.

Take moments about C, we have

W_1. AC = W.CG

i.e. W_1 (AG – CG) = W.CG

W_1.AG = (W + W_1).CG

i.e. $W_1.a$ = (W+W_1) CG

CG = $\frac{W_1 a}{W+W_1}$ ……………. (1)

When the weight W_2 is attached at B, there is loose contact at C. The reaction at C becomes zero. There is upward reaction R_2 about D.

Take moments about D, we get

W.GD = W_2 (GB –GD)

GD (W+W_2) = W_2.GB = W_2 .a

GD = $\frac{W_2 a}{W+W_2}$ ……….. (2)

$$\boxed{CG + GD = CD = b}$$

$$\therefore \frac{W_1 a}{W+W_1} + \frac{W_2 a}{W+W_2} = b$$

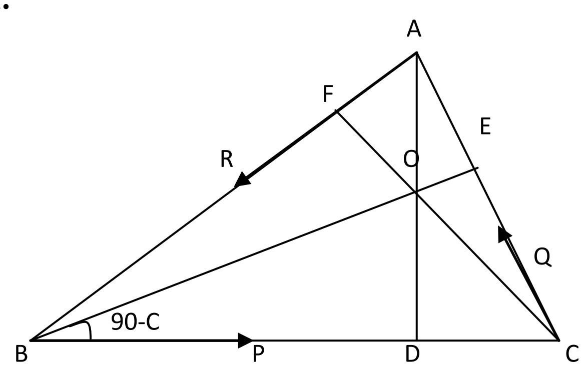
$$\frac{W_1}{W+W_1} + \frac{W_2}{W+W_2} = \frac{b}{a}$$

Problem 19

The resultant of three forces P, Q, R, acting along the sides BC, CA, AB of a triangle ABC passes through the orthocentre. Show that the triangle must be obtuse angled.

If $\angle A = 120°$, and B = C, show that Q+R = P$\sqrt{3}$.

Solution:



Let AD, BE and CF be the altitudes of the triangle intersecting at O, the orthocentre.

As the resultant passes through O, moment of the resultant about O = O.

$\therefore$ Sum of the moments of P, Q, R about O = O

P.OD+Q.OE+R.OF = 0 ……….. (1)

In rt. $\angle d\Delta BOD, \angle OBD = \angle EBC = 90° - C.$

$$\therefore \tan(90° - C) = \frac{OD}{BD}$$

i.e) cot C = $\frac{OD}{BD}$

OD = BD cot C …………. (2)

From rt. $\angle d\Delta ABD, \cos B = \frac{BD}{AB}$

$$\therefore From(2), OD = c\cos B.\cot C \;=\; c\cos B.\frac{\cos C}{\sin C}$$

$$= \frac{c}{\sin C}.\cos B\cos C$$

$= 2R'\cos B\cos C(\because \frac{c}{\sin C} = 2R'$, R' is the circumradius of the Δ)

Similarly OE = $2R'\cos C\cos A$

and OF = $2R'\cos A\cos B$

Hence (1) becomes

$$P.2R'\cos B\cos C + Q.2R'\cos C\cos A + R.2R'\cos A\cos B = 0$$

Dividing by $2R'\cos A\cos B\cos C,$

$$\frac{P}{\cos A} + \frac{Q}{\cos B} + \frac{R}{\cos C} = 0 \;\; …… (3)$$

Now, P, Q, R being magnitudes of the forces, are all positive.

(3) may hold good, if at least one of the terms must be negative.

Hence one of the cosines must be negative.

i.e) the triangle must be obtuse angled.

If A = $120°$ and the other angles equal, then B = C = $30°$

Hence (3) becomes

$$\frac{P}{\cos 120°}+\frac{Q}{\cos 30°}+\frac{R}{\cos 30°}=0$$

i.e. $$\frac{P}{\left(-\frac{1}{2}\right)}+\frac{Q+R}{\left(\frac{\sqrt{3}}{2}\right)}=0$$

i.e. $P\sqrt{3}=Q+R$

1.6 Couples: Definition

Two equal and unlike parallel forces not acting at the same point are said to constitute a couple.

Examples of a couple are the forces used in winding a clock or turning tap. Such forces acting upon a rigid body can have only a rotator effect on the body and they can not produce a motion of translation.

The moment of a couple is the product of either of the two forces of the couple and the perpendicular distance between them,

The perpendicular distance (p) between the two equal forces P of a couple is called the **arm of the couple.** A couple each of whose forces is P and whose arm is p is usually denoted by (P, p).

A couple is positive when its moment is positive i.e., if the forces of the couple tend to produce rotation in the anti-clockwise direction and a couple is negative when the forces tend to produce rotation in the clockwise direction.

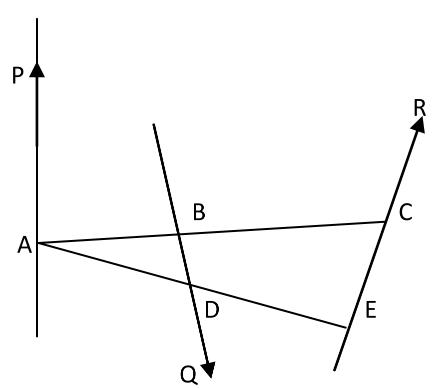
1.7 Equilibrium of three forces acting on a Rigid Body.

In the previous sections we have studied theorems and problems involving parallel forces and forces acting at a point. Here we study three important theorems and solved problems on forces acting on a rigid body and their conditions of equilibrium.

Theorem

If three forces acting on a rigid body are in equilibrium, they must be coplanar.

Proof:



Let the three forces be P,Q,R

Given : They are acting on a rigid body and in equilibrium.

Take 'A' on the force P, and B on the force Q such that AB is not parallel to R.

$\therefore$ Sum of the moments of P, Q, R about AB = 0 [$\therefore$ P,Q, R are in equilibrium]

Now, moment of P and Q about AB = 0 [$\because$ P and Q intersect AB].

$\therefore$ Moment of R about AB = 0, Hence R must intersect AB at a point C

Similarly if D is another point on Q such that AD is not parallel to R, we prove, R must intersect AD at a point E.

Since BC and DE intersect at A, BD, CE, A lie on the same plane. i.e) 'A' lies on the plane formed by Q and R. Since A is an arbitrary point on the force P, every point on the force P lie on the same plane.

ie) P, Q, R lie on the same plane.

Three Coplanar Forces – theorem

If three coplanar forces acting on a rigid body keep it in equilibrium, they must be either concurrent or all parallel.

Proof:

Let P, Q, R be the three forces acting on a rigid body keep it in equilibrium.

$\therefore$ One force must be equal and opposite to the resultant of the other two.

$\therefore$ they must be parallel or intersect.

Case 1: If P and Q are parallel (like or unlike)

Then the resultant of P and Q is also parallel. Hence R must be parallel to P and Q.

Case 2: If P and Q are not parallel: (intersect)

They meet at O. Therefore, by parallelogram law, the third force R must pass through O.

i.e) the three forces are concurrent.

Note: A couple and a single force can not be in equilibrium

Conditions of equilibrium

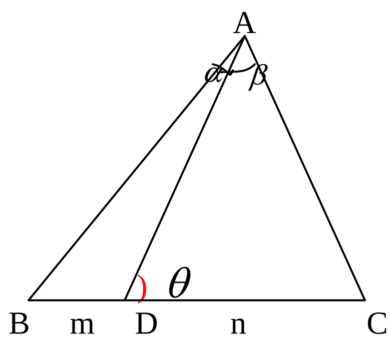
1. If three forces acting at a point are in equilibrium, then each force is proportional to the sine of the angle between the other two.
2. If three forces in equilibrium are parallel, then each force is proportional to the distance between the other two

Two Trigonometrical theorems

If D is any point on BC of a triangle ABC such that $\frac{BD}{DC}=\frac{m}{n}$ and $\angle ADC=\theta$, $\angle BAD=\alpha, \angle DAC=\beta$ then

1) $(m+n)\cot\theta = m.\cot\alpha - n.\cot\beta$ 2) $(m+n)\cot\theta = n.\cot B - m.\cot C$.

Proof:



1. Given, $\frac{m}{n}=\frac{BD}{DC}=\frac{BD}{DA}.\frac{DA}{DC}$

Using, sine formula in Δ ABD, Δ ADC,

$$\frac{m}{n}=\frac{\sin\angle BAD}{\sin\angle ABD}\times\frac{\sin\angle ACD}{\sin\angle DAC}$$

$$\frac{m}{n}=\frac{\sin\alpha}{\sin(\theta-\alpha)}\times\frac{\sin(\theta+\beta)}{\sin\beta}$$

$$=\frac{\sin\alpha}{\sin\beta}\times\frac{(\sin\theta.\cos\beta+\cos\theta.\sin\beta)}{(\sin\theta\cos\alpha-\cos\theta.\sin\alpha)}$$

Divide by $\sin\alpha.\sin\theta.\sin\beta$

$$\frac{m}{n}=\frac{\cot\beta+\cot\theta}{\cot\alpha-\cot\theta}$$

$$\therefore m(\cot\alpha-\cot\theta)=n(\cot\beta+\cot\theta)$$

$$\boxed{(m+n)\cot\theta = m.\cot\alpha - n.\cot\beta}$$

2. $\dfrac{m}{n}=\dfrac{BD}{DA}.\dfrac{DA}{DC}$

$$=\frac{\sin\angle BAD}{\sin\angle ABD}\times\frac{\sin\angle ACD}{\sin\angle DAC}$$

$$=\frac{\sin(\theta-B).\sin C}{\sin B.\sin[180°-(\theta+C)]}=\frac{\sin C.\sin(\theta-B)}{\sin B.\sin(\theta+C)}$$

$$=\frac{\sin C\times(\sin\theta.\cos B-\cos\theta\sin B)}{\sin B(\sin C\cos\theta+\cos C\sin\theta)}$$

Divide by sin B sin C sin θ

$$\frac{m}{n}=\frac{\cot B-\cot\theta}{\cot\theta+\cot C}$$

$$\therefore m(\cot\theta+\cot C)=n(\cot B-\cot\theta)$$

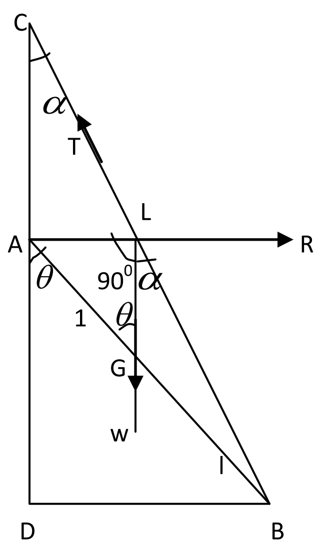
$$\boxed{\therefore(m+n)\cot\theta=n\cot B-m\cot C}$$

Problem 20

A uniform rod, of length a, hangs against a smooth vertical wall being supported by means of a string, of length *l*, tied to one end of the rod, the other end of the string being attached to a point in the wall: show that the rod can rest inclined to the wall at an angle θ given by $\cos^2\theta=\dfrac{l^2-a^2}{3a^2}$.

What are the limits of the ratio of a: *l* in order that equilibrium may be possible?

Solution:



AB is the rod of length a, with G its centre of gravity and BC is the string of length *l*. The forces acting on the rod are:

(i). Its weight W acting vertically downwards through G.

(ii). The reaction R at A which is normal to the wall and therefore horizontal.

iii) The tension T of the string along BC.

These three forces in equilibrium not being all parallel, must meet in a point L.

Let the string make an angle α with the vertical.

$\therefore \angle ACB = \alpha = \angle GLB.$

$\angle LGB = 180° - \theta \ and \angle ALG = 90°$, AG:GB = 1 :1,

Using the trigonometrical theorem in Δ ALB

$(1+1)\cot(180° - \theta) = 1.\cot 90° - 1.\cot\alpha$

i.e) $-2\cot\theta = -\cot\alpha$

$2\cot\theta = \cot\alpha$ ……………… (1)

Draw BD $\perp$ to CA.

From rt. $\angle d\Delta CDB, BD = BC.\sin\alpha = l.\sin\alpha$

rt. $\angle d\Delta ABD, BD = AB\sin\theta = a\sin\theta$

$\therefore l\sin\alpha = a\sin\theta$ ………… (2)

Eliminate α between (1) and (2).

We know that $\cos ec^2\alpha = 1 + \cot^2\alpha$ …………………… (3)

(2) $\Rightarrow \sin\alpha = \dfrac{a\sin\theta}{l}$ $\therefore \cos ec\alpha = \dfrac{l}{a\sin\theta}$ …………………… (4)

Substitute (4) and (1) in (3)

$$\frac{l^2}{a^2\sin^2\theta} = 1 + 4\cot^2\theta$$

i.e. $\dfrac{l^2}{a^2} = \sin^2\theta + 4\cos^2\theta = 1 + 3\cos^2\theta$

$$\therefore 3\cos^2\theta = \frac{l^2}{a^2} - 1 = \frac{l^2 - a^2}{a^2}$$

$\therefore \cos^2\theta = \dfrac{l^2 - a^2}{3a^2}$ ………………… (5)

Equilibrium position is possible, if $\cos^2\theta$ positive and less than 1

$\therefore l^2 - a^2 > 0$ i.e. $l^2 > a^2 or a^2 < l^2$ ……………….. (6)

Also $\dfrac{l^2 - a^2}{3a^2} < 1$ i.e. $l^2 - a^2 < 3a^2 or\ l^2 < 4a^2$

i.e. $a^2 > \dfrac{l^2}{4}$ …………………… (7)

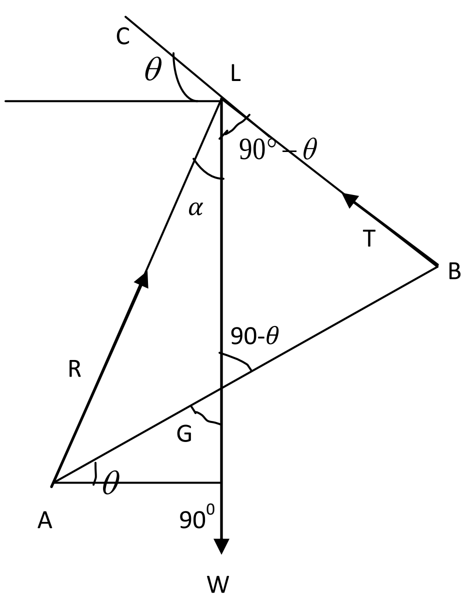
$\dfrac{l^2}{4} < a^2 < l^2$

[By (6) & (7)] $\dfrac{1}{4} < \dfrac{a^2}{l^2} < 1 = \dfrac{1}{2} < \dfrac{a}{l} < 1.$

Problem 21

A beam of weight W hinged at one end is supported at the other end by a string so that the beam and the string are in a vertical plane and make the same angle θ with the horizon. Show that the reaction at the hinge is $\dfrac{W}{4}\sqrt{8 + \operatorname{cosec}^2\theta}$

Solution:



Let AB be the beam of weight W and G its centre of gravity.

BC is the string

The force acting on the beam are:

i) Its wt. W acting vertically down wards at G

ii) the tension T along BC

iii) the reaction R at the hinge A.

For equilibrium (i) , (ii) and (iii) must meet at L.

BC and AB make the same angle θ with the horizon.

$\therefore$ They make $90^\circ - \theta$ with the vertical LG,

i.e. $\angle BLG = 90^\circ - \theta = \angle LGB$

Let $\angle ALG = \alpha$

Using trigonometrical theorem in Δ ALB, AG:GB = 1:1

$$(1+1)\cot(90^\circ - \theta) = 1.\cot\alpha - 1.\cot(90^\circ - \theta)$$

i.e. $2\ \tan\theta = \cot\alpha - \tan\theta$

$3\ \tan\theta = \cot\alpha$ .... (1)

Applying Lami's theorem at L,

$$\frac{R}{\sin(90^\circ - \theta)} = \frac{W}{\sin(90^\circ - \theta + \alpha)}$$

i.e. $\dfrac{R}{\cos\theta} = \dfrac{W}{\sin\left(90^\circ - \overline{\theta - \alpha}\right)} = \dfrac{W}{\cos(\theta - \alpha)}$

$$\therefore R = \frac{W\cos\theta}{\cos(\theta - \alpha)} = \frac{W\cos\theta}{\cos\theta\cos\alpha + \sin\theta\sin\alpha}$$

$$= \frac{W\cos\theta}{\sin\alpha(\cos\theta\cot\alpha + \sin\theta)}$$

$$= \frac{W\cos\theta}{\sin\alpha(\cos\theta.3\tan\theta + \sin\theta)} \qquad \text{[By (1)]}$$

$$= \frac{W\cos\theta\,\mathrm{cosec}\,\alpha}{3\sin\theta + \sin\theta} = \frac{W\cot\theta}{4}.\mathrm{cosec}\,\alpha = \frac{W}{4}\cot\theta\sqrt{1+\cot^2\alpha}$$

$$= \frac{W}{4}.\cot\theta\sqrt{1+9\tan^2\theta}$$

$$= \frac{W}{4}\sqrt{\cot^2\theta + 9} = \frac{W}{4}\sqrt{\cot^2\theta + 1 + 8}$$

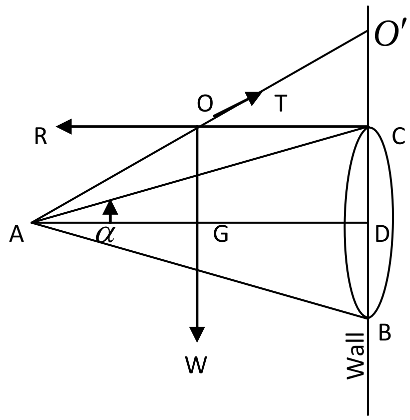
$$= \frac{W}{4}\sqrt{\mathrm{cosec}^2\theta + 8}$$

Problem 22

A solid cone of height h and semi-vertical angle α is placed with its base flatly against a smooth vertical wall and is supported by a string attached to its vertex and to a point in the wall. Show that the greatest possible length of the string is $h\sqrt{1+\frac{16}{9}\tan^2\alpha}$.

(The centre of gravity of a solid cone lies on its axis and divides it in the ratio 3 : 1 from the vertex.)

Solution:



Let A be the vertex, & height AD = h.

Semi-vertical angle $D\hat{A}C = \alpha$.

G divides AD in the ratio 3: 1

Length AO' is greatest, when the cone is just in the point of turning about C.

At that time, normal reaction R must be perpendicular to the wall.

Since, the cone is in equilibrium, the three forces T, W, R must be concurrent at O. $\Delta AOG \& \Delta AO'D$ are similar.

$$\therefore \frac{AO'}{AO} = \frac{AD}{AG} = \frac{h}{\left(\frac{3}{4}h\right)} = \frac{4}{3} \quad \boxed{\therefore AO' = \frac{4}{3}AO} \quad \text{....................} \ (1)$$

Now, OG = CD.

From $\Delta ACD, \tan\alpha = \frac{CD}{AD} = \frac{CD}{h} \quad \therefore CD = h\tan\alpha$

$$\boxed{\therefore OG = h.\tan\alpha}$$

From $\Delta AOG, AO^2 = AG^2 + GO^2$

$$= \left(\frac{3}{4}h\right)^2 + (h.\tan\alpha)^2$$

$$= \frac{9h^2}{16} + h^2.\tan^2\alpha$$

$$= \frac{9h^2 + 16h^2\tan^2\alpha}{16}$$

$$AO^2 = h^2\left(\frac{9}{16} + \tan^2\alpha\right)$$

$$\therefore AO = h.\sqrt{\frac{9}{16} + \tan^2\alpha}$$

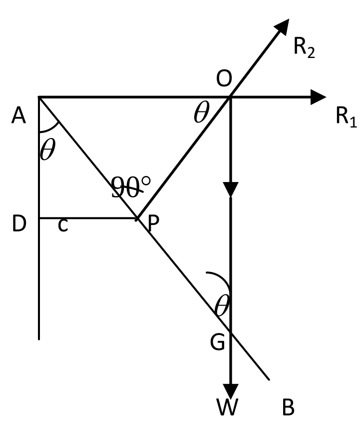
$$(1) \Rightarrow AO' = \frac{4}{3} \times h \times \sqrt{\frac{9}{16} + \tan^2\alpha}$$

$$\boxed{AO' = h.\sqrt{1 + \frac{16}{9}\tan^2\alpha}}$$

Problem 23

A heavy uniform rod of length 2a lies over a smooth peg with one end resting on a smooth vertical wall. If c is the distance of the peg from the wall and θ the inclination of the rod to the wall, show that c = a $\sin^3\theta$

Solution:



Forces acting on the rod AB are

i) Weight W at G $(\downarrow)$

ii) Reaction R_1 at A ($\perp$ to the wall)

iii) Reaction R_2 at the peg P ($\perp$ to the rod)

For equilibrium, W, R_1, R_2 must be concurrent at O.

From rightangled triangle ADP (DP = c)

$$\sin\theta = \frac{c}{AP} \text{ ………………….} \quad (1)$$

From $\Delta AOP, \sin\theta = \dfrac{AP}{AO}$ ………………….. (2)

From $\Delta OGA, \sin\theta = \dfrac{OA}{AG}$ ………………….. (3)

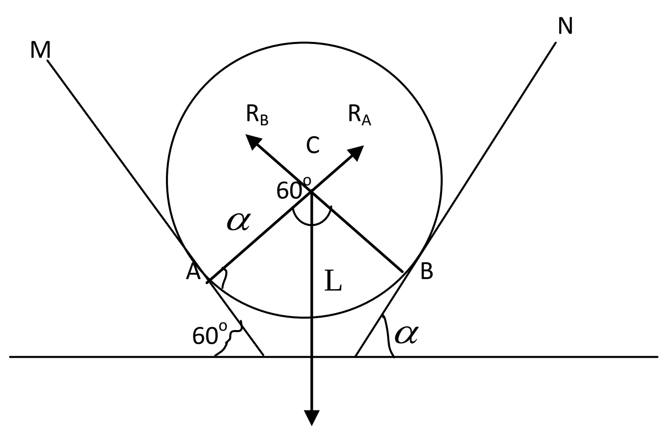
$$(1)\times(2)\times(3) \Rightarrow \sin^3\theta = \frac{c}{AP}\times\frac{AP}{AO}\times\frac{OA}{AG} = \frac{c}{AG} = \frac{c}{a}$$

$\therefore$ $\boxed{c = a\sin^3\theta}$

Problem 24

A heavy uniform sphere rests touching two smooth inclined planes one of which is inclined at 60° to the horizontal. If the pressure on this plane is one-half of the weight of the sphere, prove that the inclination of the other plane to the horizontal is 30°

Solution:



Let the sphere centre C rest on the inclined planes AM and BN. MA makes $60°$ with the horizontal and let NB make an angle α with the horizon.

The forces acting are

i) Reaction R_A at A perpendicular to the inclined plane AM and to the sphere and hence passing through C.

ii) Reaction R_B at B which is normal to the inclined plane BN and to the sphere and hence passing through C.

iii) W, the weight of the sphere acting vertically downwards at C along CL.

Clearly the above three forces meet at C.

Also $\angle ACL = 60° and \ \angle BCL = \alpha$

Applying Lami's theorem,

$$\frac{R_A}{\sin\alpha} = \frac{W}{\sin(60+\alpha)}$$

$$\therefore R_A = \frac{W\sin\alpha}{\sin(60°+\alpha)} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

But $R_A = \frac{W}{2}$ ……………… (2)

From (1) and (2), we have

$$\frac{W\sin\alpha}{\sin(60°+\alpha)} = \frac{W}{2}$$

i.e. 2 sin $\alpha = \sin(60°+\alpha) = \sin 60°\cos\alpha + \cos 60°\sin\alpha$

i.e. 2 sin $\alpha = \frac{\sqrt{3}}{2}\cos\alpha + \frac{1}{2}\sin\alpha$ or $4\sin\alpha = \sqrt{3}\cos\alpha + \sin\alpha$

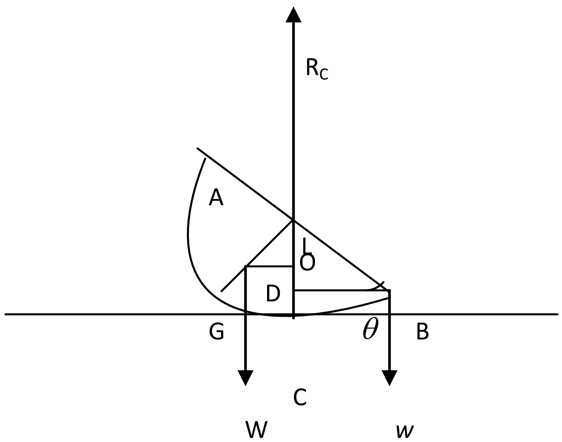
i.e. 3 sin $\alpha = \sqrt{3}\cos\alpha$ or $\frac{\sin\alpha}{\cos\alpha} = \frac{\sqrt{3}}{3} = \frac{1}{\sqrt{3}}$

i.e. tan $\alpha = \frac{1}{\sqrt{3}}$ or $\alpha = 30°$

Problem 25

A uniform solid hemisphere of weight W rests with its curved surface on a smooth horizontal plane. A weight w is suspended from a point on the rim of the hemisphere. If the plane base of the rim is inclined to the horizontal at an angle θ, prove that $\tan\theta = \dfrac{8w}{3W}$

Solution:



Draw GL perpendicular to OC and BD perpendicular to OC. Base AB is inclined at an angle θ with the horizontal BD. Forces acting are i) Reaction R_C ii) Weight W at G iii) Weight w at B.

Since these three forces are parallel, and in equilibrium each force is proportional to the distance between the other two.

$$\therefore \frac{W}{BD} = \frac{w}{GL} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ (1)$$

Now, $\Delta OBD \Rightarrow BD = OB\cos\theta = r\cos\theta$

Here, OG = $\dfrac{3r}{8}$, r – radius

$$GL = OG.\sin\theta = \frac{3r}{8}\sin\theta$$

$$\therefore (1) \Rightarrow \frac{W}{r\cos\theta} = \frac{w}{\left(\frac{3r}{8}\sin\theta\right)}$$

$$\boxed{\therefore \tan\theta = \frac{8w}{3W}}$$

UNIT II

2.1 Friction

In the previous sections we have studied problems on equilibrium of smooth bodies. Practically no bodies are perfectly smooth. All bodies are rough to a certain extent. Friction is the force that opposes the motion of an object. Only because of this friction we are able to travel along the road by walking or by vehicles. So friction helps motion. It is a tangential force acting at the point on contact of two bodies. To stop a moving object a force must act in the opposite direction to the direction of motion. Such force is called a frictional force. For example if you push your book across your desk, the book will move. The force of the push moves the book. As the books slides across the desk, it slows down and stops moving. When you ride a bicycle the contact between the wheel and the road is an example of dynamic friction.

Definition

If two bodies are in contact with one another, the property of the two bodies, by means of which a force is exerted between them at their point of contact to prevent one body from sliding on the other, is called ***friction;*** the force exerted is called the ***force of friction***.

Types of Friction

There are three types of friction

1) Statical Friction 2) Limiting Friction 3) Dynamical friction.

1. When one body in contact with another is in equilibrium, the friction exerted is just sufficient to maintain equilibrium is called ***statical friction***.

2. When one body is just on the point of sliding on another, the friction exerted attains its maximum value and is called ***limiting friction***; the equilibrium is said to be limiting equilibrium.

3. When motion ensues by one body sliding over another, the friction exerted is called ***dynamical friction***.

2.2 Laws of Friction

Friction is not a mathematical concept; it is a physical reality.

Law 1 When two bodies are in contact, the direction of friction on one of them at the point of contact is opposite to the direction in which the point of contact would commence to move.

Law 2 When there is equilibrium, the magnitude of friction is just sufficient to prevent the body from moving.

Law 3 The magnitude of the limiting friction always bears a constant ratio to the normal reaction and this ratio depends only on the substances of which the bodies are composed.

Law 4 The limiting friction is independent of the extent and shape of the surfaces in contact, so long as the normal reaction is unaltered.

Law 5 (Law of dynamical Friction)

When motion ensues by one body sliding over the other the direction of friction is opposite to that of motion; the magnitude of the friction is independent of the velocity of the point of contact but the ratio of the friction to the normal reaction is slightly less when the body moves, than when it is in limiting equilibrium.

Friction is a passive force: Explain

1) Friction is only a resisting force.
2) It appears only when necessary to prevent or oppose the motion of the point of contact.
3) It can not produce motion of a body by itself, but maintains relative equilibrium.
4) It is a self-adjusting force.
5) It assumes magnitude and direction to balance other forces acting on the body.

Hence, friction is purely a passive force.

Co-efficient of friction

The ratio of the limiting friction to the normal reaction is called the co-efficient of friction. It is denoted by μ

i.e.) $\boxed{\frac{F}{R} = \mu}$ $\Rightarrow$ $\boxed{F = \mu R}$

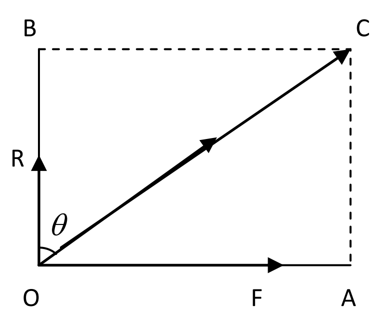
Note: 1) μ depends on the nature of the materials in contact.

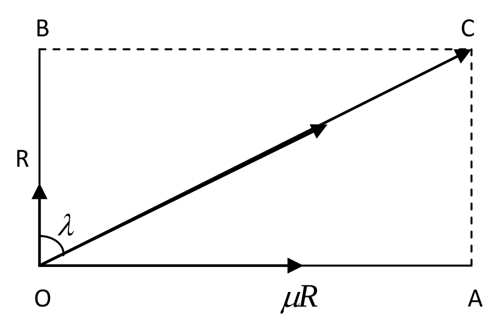
2) Friction is maximum when it is limiting. μR is the maximum value of friction.

3) When equilibrium is non-limiting, $F < \mu R$ i.e.) $\frac{F}{R} < \mu$

4) Friction 'F' takes any value from zero upto μR.

Angle of Friction





Let OA = F(Friction), $\overrightarrow{OB} = R$ (Normal reaction) & $\overrightarrow{OC}$ be the resultant of F and R.

If $B\hat{O}C = \theta$, $\tan\theta = \frac{BC}{OB} = \frac{OA}{OB} = \frac{F}{R}$ ……….. (1)

As F increases, θ - increases until F reaches its maximum value μR. In this case, equilibrium is limiting.

Definition

"When one body is in limiting equilibrium over another, the angle which the resultant reaction makes with the normal at the point of contact is called **the angle of friction** and is denoted by λ"

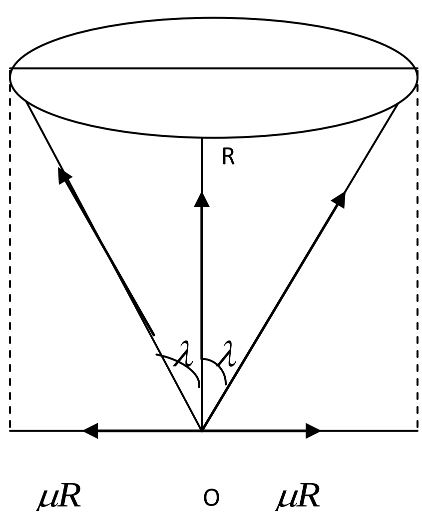
In the limiting equilibrium, $B\hat{O}C = \lambda$ = angle of friction.

$\therefore \tan\lambda = \frac{BC}{OB} = \frac{OA}{OB} = \frac{\mu R}{R} = \mu$

$$\boxed{\mu = \tan\lambda}$$

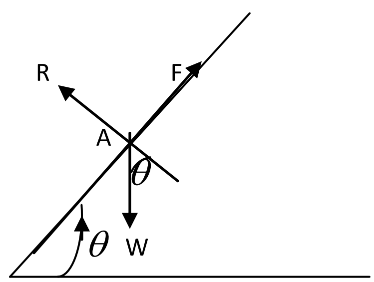
i.e.) The co-efficient of friction is equal to the tangent of the angle of friction.

Cone of Friction



We know, the greatest angle made by the resultant reaction with the normal is λ (angle of friction) where $\lambda = \tan^{-1}(\mu)$. Consider the motion of a body at O (its point of contact) with another. When two bodies are in contact, consider a cone drawn with O as vertex, common normal as the axis of the cone, λ - be the semi-vertical angle of the cone. Now, the resultant reaction of R and μR will have a direction which lies within the surface or on the surface of the cone. It can not fall outside the cone. This cone generated by the resultant reaction is called the ***cone of friction***.

2.3 Equilibrium of a particle on a rough inclined plane.



Let θ - be the inclination of the rough inclined plane, on which a particle of weight W, is placed at A. Forces acting on the particle are,

1) Weight W vertically downwards
2) Normal reaction R, $\perp$r to the plane.
3) Frictional force F, along the plane upwards (Since the body tries to slip down).

Resolving the forces along and perpendicular to the plane,

F = $W \sin\theta, \quad R = W\cos\theta$

$$\therefore \frac{F}{R} = \tan\theta$$

But $\frac{F}{R} < \mu \ \therefore \tan\theta < \mu$

i.e) $\tan\theta < \tan\lambda$

$\therefore \theta < \lambda$

When $\theta = \lambda, \frac{F}{R} = \tan\lambda = \mu$

Hence, it is clear that "***when a body is placed on a rough inclined plane and is on the point of sliding down the plane, the angle of inclination of the plane is equal to the angle of friction***."

Now λ is called as the ***angle of repose.***

Thus the angle of repose of a rough inclined plane is equal to the angle friction when there is no external force act on the body.

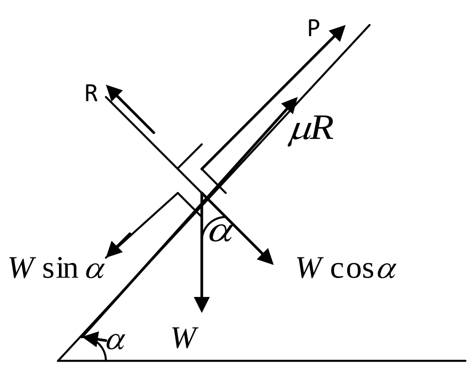
2.4 Equilibrium of a body on a rough inclined plane under a force parallel to the plane.

A body is at rest on a rough plane inclined to the horizon at an angle greater than the angle of friction and is acted on by a force parallel to the plane. Find the limits between which the force must lie.

Proof:

Let α be the inclination of the plane, W be the weight of the body& R be the normal reaction.

Case 1: Let the body be on the point of slipping down. Therefore μR acts upwards along the plane.



Let P be the force applied to keep the body at rest.

Resolving the forces along and perpendicular to the plane,

$P + \mu R = W \sin \alpha$ ……………….. (1)

$R = W.\cos \alpha$ ………………… (2)

$\therefore P = W.\sin \alpha - \mu.W \cos\alpha$

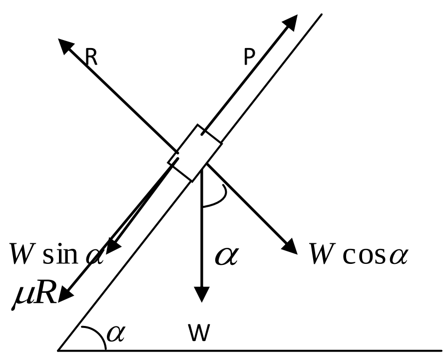
$= W[\sin \alpha - \tan \lambda.\cos \alpha]$

$= \dfrac{W}{\cos \lambda}[\sin \alpha.\cos \lambda - \cos \alpha \sin \lambda]$

$= \dfrac{W}{\cos \lambda}.\sin(\alpha - \lambda)$

Let $P_1 = \dfrac{W.\sin(\alpha - \lambda)}{\cos \lambda}$

Case ii Let the body be on the point of moving up. Therefore limiting frictional force μR acts downward along the plane.



Let P be the external force applied to keep the body at rest.

Resolving the force,

$R = W \cos \alpha; \; P = \mu R + W \sin \alpha$

$\therefore P = \mu.W \cos \alpha + W \sin \alpha$

$= \dfrac{W}{\cos \lambda}[\sin \lambda \cos \alpha + \cos \lambda.\sin \alpha]$

$= \dfrac{W}{\cos\lambda}.\sin(\alpha+\lambda)$

Let $P_2 = \dfrac{W}{\cos\lambda}.\sin(\alpha+\lambda)$

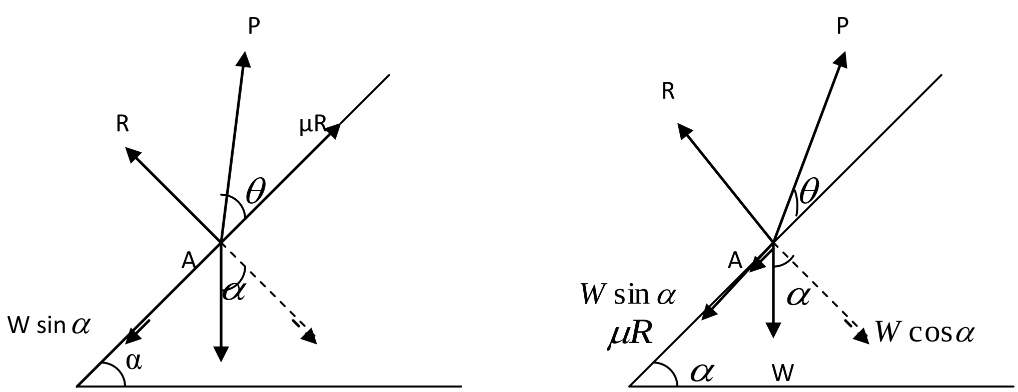
If $P < P_1$, body will move down the plane. If $P > P_2$, body will move up the plane.

$\therefore$ For equilibrium P must lie between P_1 and P_2.

i.e.) $\boxed{P_1 > P > P_2}$

2.5 Equilibrium of a body on a rough inclined plane under any force.

Theorem: A body is at rest on a rough inclined plane of inclination α to the horizon, being acted on by a force making an angle θ with the plane; to find the limits between which the force must lie and also to find the magnitude and direction of the least force required to drag the body up the inclined plane.



Let α be the inclination of the plane, W be the weight of the body, P – be the force acting at an angle θ with the inclined plane and R – be the normal reaction.

Case i: The body is just on the point of slipping down. Therefore the limiting friction μR acts upwards.

Resolving the forces along and $\perp r$ to the inclined plane,

$P\cos\theta + \mu R = W\sin\alpha$ ……………….. (1)

$P\sin\theta + R = W\cos\alpha$ ………………….. (2)

$$\therefore R = W\cos\alpha - P\sin\theta$$

$$\therefore (1) \Rightarrow P\cos\theta + \mu(W\cos\alpha - P\sin\theta) = W\sin\alpha$$

$$P(\cos\theta - \mu\sin\theta) = W(\sin\alpha - \mu\cos\alpha)$$

$$\therefore P = \frac{W(\sin\alpha - \mu\cos\alpha)}{\cos\theta - \mu\sin\theta}$$

We have $\mu = \tan\lambda$

$$\therefore P = \frac{W(\sin\alpha - \tan\lambda.\cos\alpha)}{\cos\theta - \tan\lambda.\sin\theta}$$

$$= W\frac{(\sin\alpha\cos\lambda - \cos\alpha.\sin\lambda)}{\cos\theta.\cos\lambda - \sin\theta.\sin\lambda}$$

$$= W\frac{\sin(\alpha - \lambda)}{\cos(\theta + \lambda)}$$

Let $P_1 = W.\dfrac{\sin(\alpha - \lambda)}{\cos(\theta + \lambda)}$

Case ii: The body is just on the point of moving up the plane. Therefore μR acts downwards.

Resolving the forces along and $\perp r$ to the plane.

$P\cos\theta - \mu R = W.\sin\alpha$ …………………. (3)

$P\sin\theta + R = W.\cos\alpha$ …………………. (4)

$$R = W\cos\alpha - P\sin\theta$$

$$(3) \Rightarrow P\cos\theta - \mu(W\cos\alpha - P\sin\theta) = W.\sin\alpha$$

$$P(\cos\theta + \mu\sin\theta) = W(\sin\alpha + \mu\cos\alpha)$$

$$\therefore P = \frac{W(\sin\alpha + \tan\lambda.\cos\alpha)}{(\cos\theta + \tan\lambda.\sin\theta)}$$

$$= \frac{W(\sin\alpha.\cos\lambda + \sin\lambda.\cos\alpha)}{(\cos\theta\cos\lambda + \sin\theta.\sin\lambda)}$$

$$= \frac{W.\sin(\alpha + \lambda)}{\cos(\theta - \lambda)}$$

Let $P_2 = \dfrac{W.\sin(\alpha+\lambda)}{\cos(\theta-\lambda)}$

To keep the body in equilibrium, P_1 and P_2 are the limiting values of P.

Find the least force required to drag the body up the inclined plane

We have, P = $W.\dfrac{\sin(\alpha+\lambda)}{\cos(\theta-\lambda)}$

P is least when $\cos(\theta-\lambda)$ is greatest.

i.e.) When $\cos(\theta-\lambda)=1$

i.e.) When $\theta-\lambda=0$

i.e.) When $\theta=\lambda$

$$\boxed{\therefore Least\ \ value\ \ of\ \ P = W.\sin(\alpha+\lambda)}$$

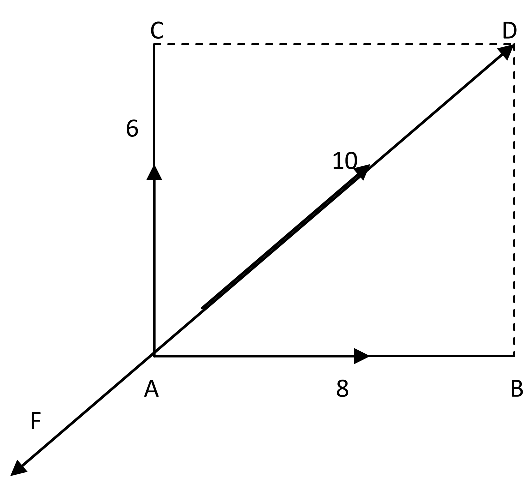
Hence the force required to move the body up the plane will be least when it is applied in a direction making with the inclined plane an angle equal to the angle of friction.

i.e.) ***"The best angle of traction up a rough inclined plane is the angle of friction"***

Problem 1

A particle of weight 30 kgs. resting on a rough horizontal plane is just on the point motion when acted on by horizontal forces of 6kg wt. and 8kg. wt. at right angles to each other. Find the coefficient of friction between the particle and the plane and the direction in which the friction acts.

Solution:



Let AB = 8 and AC = 6 represent the directions of the forces, A being the particle.

The resultant force = $\sqrt{8^2+6^2}$ = 10kg. wt. and this acts along AD, making an angle $\cos^{-1}\left(\frac{4}{5}\right)$ with the 8kg force.

Let F be the frictional force. As motion just begins, magnitude of F is equal to that of the resultant force.

$\therefore F = 10$ ……………… (1)

If R is the normal reaction on the particle,

R = 30 ………………….. (2)

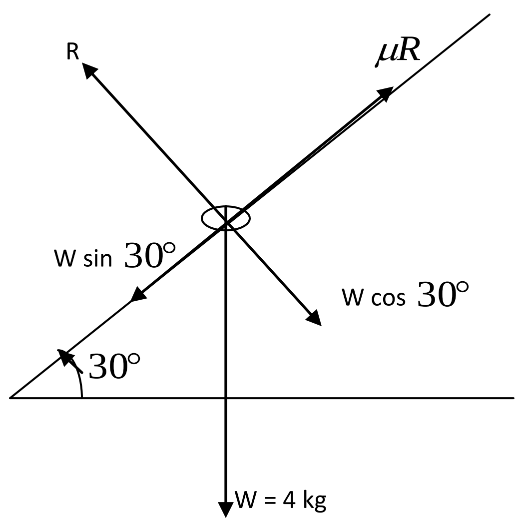
If μ is the coefficient of friction as the equilibrium is limiting, $F = \mu R$

10 = $\mu.30$ $\therefore \mu = \frac{10}{30} = \frac{1}{3}$.

Problem 2

A body of weight 4 kgs. rests in limiting equilibrium on an inclined plane whose inclination is 30°. Find the coefficient of friction and the normal reaction.

Solution:



Since the body is in limiting equilibrium on the inclined plane, it tries to move in the downward direction along the inclined plane.

$\therefore$ Frictional force μR acts in the upward direction along the inclined plane. Resolving along and $\perp r$ to the plane,

$\mu R = W \sin 30°$ ………………… (1)

$= 4.\frac{\sqrt{3}}{2} = 2\sqrt{3}$

R = $W.\cos 30°$ …………………. (2)

$= 4\frac{1}{2} = 2$

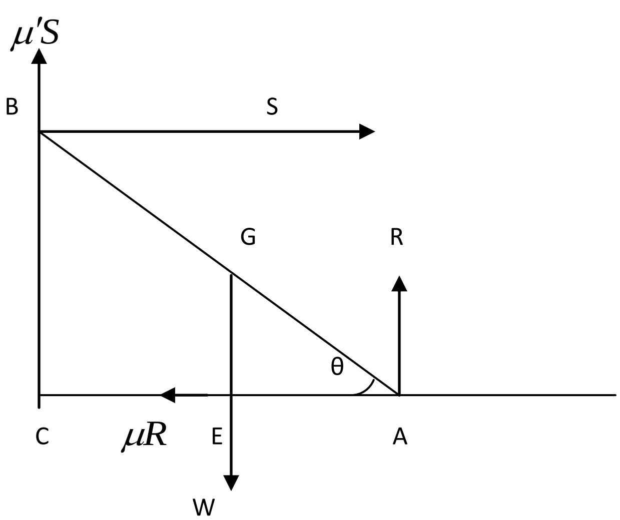
$\frac{(1)}{(2)} \Rightarrow \mu = \frac{1}{\sqrt{3}}$

$\tan\lambda = \frac{1}{\sqrt{3}}$, $\therefore$ $\boxed{\lambda = 30°}$

Problem 3

A uniform ladder is in equilibrium with one end resting on the ground and the other against a vertical wall; if the ground and wall be both rough, the coefficients of friction being μ and μ' respectively, and if the ladder be on the point of slipping at both ends, show that θ, the inclination of the ladder to the horizon is given by $\tan\theta = \frac{1-\mu\mu'}{2\mu}$. Find also the reactions at the wall and ground.

Solution:



AB is the uniform ladder, whose weight W is acting at G such that AG = GB. Forces acting are,

1. Weight W
2. Normal reaction R at A
3. Normal reaction S at B
4. μR
5. $\mu' S$

When the ladder is on the point of slipping at both ends, frictional forces $\mu' S, \mu R$ act along CB, AC respectively.

Since the ladder is in equilibrium resultant is zero.

$\therefore$ Resolving horizontally and vertically,

$S = \mu R$ ……….. (1)

$R + \mu' S = W$ …………. (2)

$\therefore R + \mu'(\mu R) = W$

$R(1+\mu\mu') = W \Rightarrow$ $\boxed{R = \dfrac{W}{1+\mu\mu'}}$ $\boxed{\therefore S = \dfrac{\mu W}{1+\mu\mu'}}$

By Varigon's theorem on moments, taking moments about A

$$S.BC + \mu' S.AC = W.AE$$

$$S.AB\sin\theta + \mu' S.AB\cos\theta = W.AG.\cos\theta$$

$$S.\sin\theta + \mu' S.\cos\theta = W.\frac{1}{2}.\cos\theta \quad \left[\because AG = \frac{AB}{2}\right]$$

$$\therefore S.\sin\theta = \left[\frac{W}{2} - \mu' S\right].\cos\theta$$

$$\therefore \tan\theta = \frac{W}{2S} - \mu' = \frac{W}{2\left[\dfrac{\mu W}{1+\mu\mu^1}\right]} - \mu^1 = \frac{1+\mu\mu'}{2\mu} - \mu'$$

$$= \frac{1+\mu\mu' - 2\mu\mu'}{2\mu} \quad \boxed{\tan\theta = \frac{1-\mu\mu'}{2\mu}}$$

Problem 4

In the previous problem, when $\mu = \mu'$ show that $\theta = 90° - 2\lambda,$ where λ is the angle of friction.

Solution:

In the previous problem, we have proved $\tan\theta = \dfrac{1-\mu\mu'}{2\mu}$

Put $\mu = \mu'$, we get

$$\tan\theta = \frac{1-\mu^2}{2\mu} = \frac{1-\tan^2\lambda}{2\tan\lambda};[\because \mu = \tan\lambda]$$

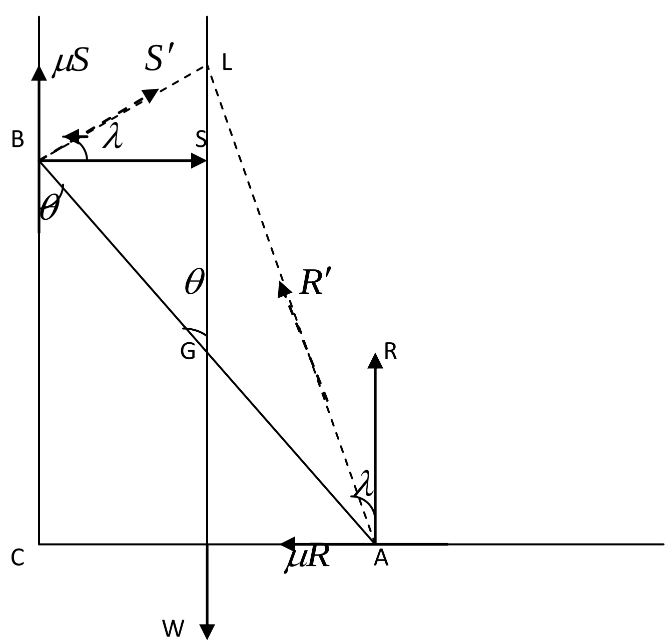
$$= \frac{1}{\tan 2\lambda} = \cot 2\lambda = \tan(90° - 2\lambda)$$

i.e.) $\tan\theta = \tan(90° - 2\lambda)$ $\therefore$ $\boxed{\theta = 90° - 2\lambda}$

Problem 5

A uniform ladder rests in limiting equilibrium with its lower end on a rough horizontal plane and its upper end against an equally rough vertical wall. If θ be the inclination of the ladder to the vertical, prove that $\tan\theta = \dfrac{2\mu}{1-\mu^2}$ where μ is the coefficient of friction.

Solution:



When the ladder AB is in limiting equilibrium, five forces are acting as marked in the figure.

1) Weight of the ladder W
2) Normal reaction R at A
3) Normal reaction S at B
4) Frictional force μR
5) frictional force μS

Let R', S' be the resultant reactions of R, μR and S, μS respectively.

$\therefore$ We have 3 forces R', S', W . For equilibrium, they must be concurrent at L.

In $\Delta LAB, L\hat{G}A = 180° - \theta; A\hat{L}G = \lambda$

$B\hat{L}G = 90 - \lambda, AG : GB = 1:1$

$\therefore$ By trigonometrical theorem in Δ LBA,

(1+1) $\cot(180° - \theta) = 1.\cot(90° - \lambda) - 1.\cot\lambda$

$$-2.\cot\theta = \tan\lambda - \cot\lambda \;\; = \frac{\tan^2\lambda - 1}{\tan\lambda}$$

$$\therefore \cot\theta = \frac{1 - \tan^2\lambda}{2\tan\lambda}$$

i.e.) $\dfrac{1}{\tan\theta} = \dfrac{1-\mu^2}{2\mu}$ $\quad \boxed{\therefore \tan\theta = \dfrac{2\mu}{1-\mu^2}}$

Problem 6

A uniform ladder rests with its lower end on a rough horizontal ground its upper end against a rough vertical wall, the ground and the wall being equally rough and the angle of friction being λ. Show that the greatest inclination of the ladder to the vertical is 2λ .

Solution

In the previous problem, we have proved, $\tan\theta = \dfrac{2\mu}{1-\mu^2}$ But $\mu = \tan\lambda$

$$\therefore \tan\theta = \frac{2\tan\lambda}{1-\tan^2\lambda} = \tan 2\lambda \Rightarrow \boxed{\therefore \theta = 2\lambda}$$

Problem 7

A ladder which stands on a horizontal ground, leaning against a vertical wall, is so loaded that its C. G. is at a distance a and b from its lower and upper ends respectively. Show that if the ladder is in limiting equilibrium, its inclination θ to the horizontal is given by $\tan\theta = \dfrac{a - b\mu\mu'}{(a+b)\mu}$ where μ, μ' are the coefficients of friction between the ladder and the ground and the wall respectively.

Solution:

As in problem 5, five forces are acting on the ladder

Here, AG : GB = a: b

$\therefore$ By Trigonometrical theorem in ΔLBA,

$(b+a).\cot(90+\theta) = b.\cot(90-\lambda') - a.\cot\lambda$

i.e.) $(a+b)(-\tan\theta) = b.\tan\lambda^1 - a.\cot\lambda$

$$\therefore \tan\theta = \frac{\left(\dfrac{a}{\mu}\right) - b.\mu'}{a+b} = \frac{a - b.\mu\mu'}{(a+b)\mu}$$

Problem 8

A ladder AB rests with A on a rough horizontal ground and B against an equally rough vertical wall. The centre of gravity of the ladder divides AB in the ratio a: b. If the ladder is on the point of slipping, show that the inclination θ of the ladder to the ground is given by $\tan\theta = \dfrac{a - b\mu^2}{\mu(a+b)}$ where μ is the coefficient of friction.

Solution:

In the previous problem,

Put $\mu = \mu'$ in $\tan\theta = \dfrac{a - b\mu\mu'}{(a+b)\mu}$ $\boxed{\therefore \tan\theta = \dfrac{a - b\mu^2}{\mu(a+b)}}$

Problem 9

A ladder AB rests with A resting on the ground and B against a vertical wall, the coefficients of friction of the ground and the wall being μ and μ' respectively. The centre of gravity G of the ladder divides AB in the ratio 1: n. If the ladder is on the point of slipping at both ends, show that its inclination to the ground is given by $\tan\theta = \dfrac{1-n\mu\mu'}{(n+1)\mu}$.

Solution:

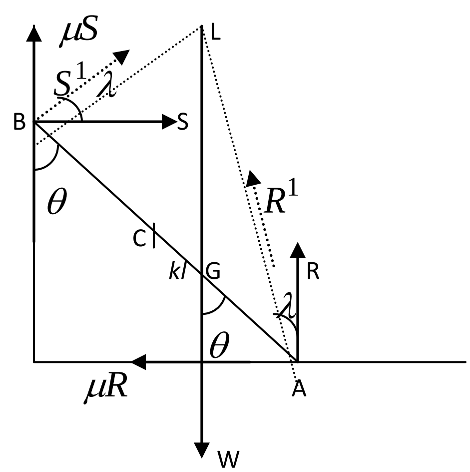
Put a : b = 1 : n in problem7.

$$\therefore \tan\theta = \frac{1-n\mu\mu'}{(1+n)\mu}$$

Problem 10

A ladder of length $2l$ is in contact with a vertical wall and a horizontal floor, the angle of friction being λ at each contact. If the weight of the ladder acts at a point distant kl below the middle point, prove that its limiting inclination θ to the vertical is given by $\cot\theta = \cot 2\lambda - k\,\mathrm{cosec}\,2\lambda.$

Solution:



Forces are acting as marked in the figure. For equilibrium, the three forces R', S', W must be concurrent at L, where W – be the weight of the ladder.

In $\Delta LAB, BC = CA = l; CG = kl.$

$$\therefore BG = BC + CG = l + kl = (1+k)l$$

$B\hat{L}G = 90^\circ - \lambda, L\hat{G}A = 180^\circ - \theta$

$A\hat{L}G = \lambda; GA = CA - CG = l - kl = (1-k)l.$

$BG : GA = (1+k) : (1-k)$

$\therefore$ By Trigonometrical theorem in $\Delta LBA,$

$[(1+k)+(1-k)].\cot(180^\circ - \theta) = (1+k).\cot(90^\circ - \lambda) - (1-k).\cot\lambda.$

$2(-\cot\theta) = (1+k).\tan\lambda - (1-k).\cot\lambda$

$\therefore 2\cot\theta = (1-k)\cot\lambda - (1+k)\tan\lambda$

$$= \frac{(1-k).\cot^2\lambda - (1+k)}{\cot\lambda}$$

$$= \frac{(\cot^2\lambda - 1) - k(\cot^2\lambda + 1)}{\cot\lambda}$$

$$\cot\theta = \frac{(\cot^2\lambda - 1) - k.\cos ec^2\lambda}{2.\cot\lambda}$$

$$= \frac{1-\tan^2\lambda}{2\cot\lambda.\tan^2\lambda} - k\left[\frac{1+\cot^2\lambda}{2.\cot\lambda}\right]$$

$$= \frac{1}{\left(\dfrac{2\tan\lambda}{1-\tan^2\lambda}\right)} - k\left[\frac{1+\tan^2\lambda}{2.\tan^2\lambda.\cot\lambda}\right]$$

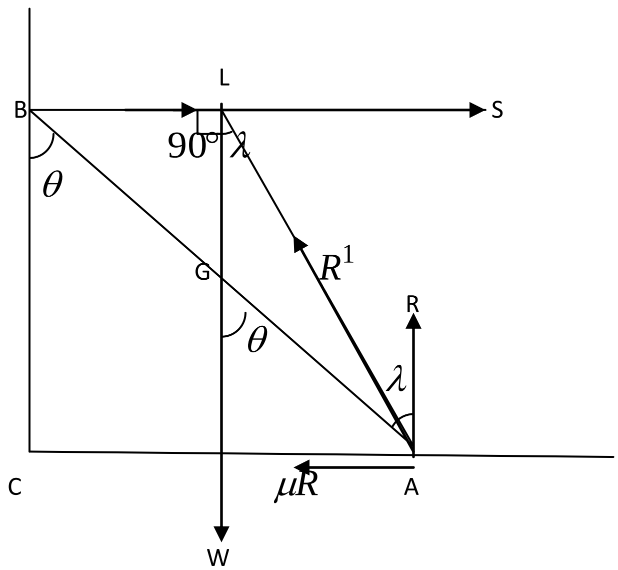
$$= \frac{1}{\tan 2\lambda} - k.\frac{1}{\sin 2\lambda}$$

ie) $\cot\theta = \cot 2\lambda - k.\cos ec 2\lambda$

Problem 11

A uniform ladder rests in limiting equilibrium with its lower end on a rough horizontal plane and with the upper end against a smooth vertical wall. If θ be the inclination of the ladder to the vertical, prove that, $\tan\theta = 2\mu,$ where μ is the coefficient of friction.

Solution:



Since the wall is smooth, there is no frictional force. Forces acting on the ladder are i) its weight W, ii) Frictional force μR iii) R at A iv) S at B. For equilibrium, the three forces W, R', S must be concurrent at L. where R^1 is the resultant of R and μR. In triangle LAB,

$L\hat{G}A = 180^\circ - \theta, A\hat{L}G = \lambda, B\hat{L}G = 90^\circ; BG : GA = 1:1. A\hat{B}C = \theta$

By Trigonometrical theorem in ΔLAB,

$(1+1)\cot(180^\circ - \theta) = 1.\cot 90^\circ - 1.\cot\lambda$

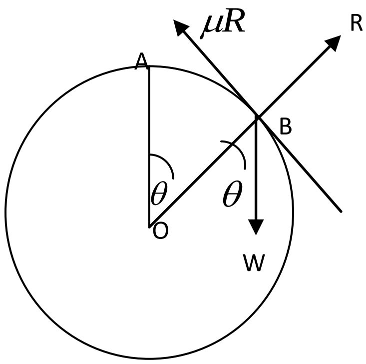
$-2.\cot\theta = 0 - \cot\lambda$

$\therefore \frac{2}{\tan\theta} = \frac{1}{\tan\lambda}$ $\therefore \tan\theta = 2\tan\lambda$ i.e) $\boxed{\tan\theta = 2\mu}$

Problem 12

A particle is placed on the outside of a rough sphere whose coefficient of friction is μ. Show that it will be on the point of motion when the radius from it to the centre makes an angle $\tan^{-1}\mu$ with the vertical.

Solution:



Let O be the centre, A the highest point of the sphere and B the position of the particle which is just on the point of motion. Let $\angle AOB = \theta$

The forces acting at B are:

1) the normal reaction R
2) limiting friction μR
3) Its weight W,

Since the particle at B is in limiting equilibrium,
Resolving along the normal OB,

$R = W\cos\theta$ …………………. (1)

Resolving along the tangent at B,

$\mu R = W\sin\theta$ ……………….. (2)

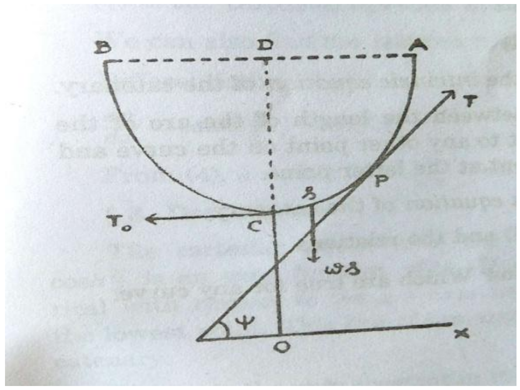
$\frac{(2)}{(1)} \Rightarrow \mu = \tan\theta \Rightarrow$ $\boxed{\theta = \tan^{-1}\mu}$

2.6 Equilibrium of Strings

When a uniform string or chain hangs freely between two points not in the same vertical line, the curve in which it hangs under the action of gravity is called a *catenary*. If the weight per unit length of the chain or string is constant, the catenary is called the *uniform or common catenary*.

2.7 Equation of the common catenary:

A uniform heavy inextensible string hangs freely under the action of gravity; to find the equation of the curve which it forms.



Let ACB be a uniform heavy flexible cord attached to two points A and B at the same level, C being the lowest, of the cord. Draw CO vertical, OX horizontal and take OX as X axis and OC as Y axis. Let P be any point of the string so that the length of the are CP = s

Let ω be the weight per unit length of the chain.

Consider the equilibrium of the portion CP of the chain.

The forces acting on it are:

(i) Tension T_0 acting along the tangent at C and which is therefore horizontal.

(ii) Tension T acting at P along the tangent at P making an angle Ψ with OX.

(iii) Its weight **ws** acting vertically downwards through the C.G. of the arc CP.

For equilibrium, these three forces must be concurrent.

Hence the line of action of the weight **ws** must pass through the point of the intersection of T and T_o.

Resolving horizontally and vertically, we have

$$T\cos\Psi = T_o \quad \dots \dots \quad (1)$$

$$\text{and } T\sin\Psi = \mathbf{ws} \quad \dots \dots \quad (2)$$

Dividing (2) by (1), $\tan\Psi = \frac{\mathbf{ws}}{T_0}$

Now it will be convenient to write the value of T_o the tension at the lowest point, as $T_o = wc$ … … (3) where *c* is a constant. This means that we assume T_o, to be equal to the weight of an unknown length *c* of the cable.

Then tan $\Psi = \frac{\boldsymbol{ws}}{\boldsymbol{wc}} = \frac{\boldsymbol{s}}{\boldsymbol{c}}$

$\therefore$ S = ctanΨ … … … (4)

Equation (4) is called the *intrinsic* equation of the catenary.

It gives the relation between the length of the area of the curve from the lowest point to any other point on the curve and the inclination of the tangent at the latter point.

To obtain the *certesian equation* of the catenary,

We use the equation (4) and the relations

$\frac{dy}{ds}$ = sin Ψ and $\frac{dy}{dx}$ = tan Ψ which are true for any curve.

Now $\frac{dy}{d\Psi} = \frac{dy}{ds} \cdot \frac{ds}{d\Psi}$

$= \sin \Psi \frac{d}{d\Psi} c \tan \Psi$

$= \sin c\sec^2\Psi = c\sec \Psi \tan \Psi$

$\therefore y = \int c\sec \Psi \tan \Psi \, d\Psi + A$

$= c\sec \Psi + S$

If $y = c$ when $\Psi = 0$, then c = $c$sec0 + A

$\therefore$ A = 0

Hence y = csec Ψ … … … (5)

$\therefore y^2 = c^2 \sec \Psi = c^2 (1 + \tan^2 \Psi)$

$= c^{\,2} + s^{\,2}$ … … (6)

$\frac{dy}{dx} = \tan \Psi = \frac{s}{c} = \frac{\sqrt{y^2 - c^2}}{\mathrm{c}}$

$\therefore \frac{\mathrm{dy}}{\sqrt{y^2 - c^2}} = \frac{\mathrm{dx}}{\mathrm{c}}$

Integrating, $\cos h^{-1} \left(\frac{y}{c}\right) = \frac{x}{c} + B$

When x = 0, y = c

i.e. $\cos h^{-1} 1 = 0 + B$ or B = 0

$\therefore \cos h^{-1} \left(\frac{y}{c}\right) = \frac{x}{c}$

i.e. $y = c\cos h \left(\frac{\mathrm{x}}{c}\right)$ … … (7)

(7) is the Cartesian equation to the catenary.

We can also find the relation connecting s and x.

Differentiating (7).

$\frac{dy}{dx}$ = csinh $\frac{x}{c}$. $\frac{1}{c}$ = sinh $\frac{x}{c}$

From (4), s = ctan Ψ = c. $\frac{dy}{dx}$ = csinh $\frac{x}{c}$ … (8)

Definitions:

The Cartesian equation to the catenary is y = ccosh $\frac{x}{c}$. cosh $\frac{x}{c}$ is an even function of x. Hence the curve is symmetrical with respect to the y-axis i.e. to the vertical through the lowest point. This line of symmetry is called the axis of the catenary.

Since c is the only constant, in the equation, it is called the *parameter* of the catenary and it determines the size of the curve.

The lowest point *C* is called the vertex of the catenary. The horizontal line at the depth *c* below the vertex (which is taken by us the x – axis) is called the directrix of the catenary.

If the two points A and B from where the string is suspended are in a horizontal line, then the distance AB is called the span and the distance CD (i.e. the depth of the lowest point C below AB) is called the sag.

2.8 Tension at any point:

We have derived the equations

T cos Ψ = T_0 … … … … (1)

And T sin Ψ = *ws* … … … … (2)

We have also put T_0 = *wc* … … … (3)

Equation (3) shows that the tension at the lowest point is a constant and is equal to the weight of a portion of the string whose length is equal to the parameter of the catenary. From the equation (1), we find that the horizontal component of the tension at any point on the curve is equal to the tension at the lowest point and hence is a constant.

From equation (2), we deduce that the vertical component of the tension at any point is equal to ws i.e. equal to the weight of the portion of the string lying between the vertex and the point. (∴ s = are CP)

Squaring (1) and (2) and then adding,

$T^2 = T^2_0 + w^2s^2$

$=w^2c^2+w^2a^2$

$=w^2(c^2+s^2)$

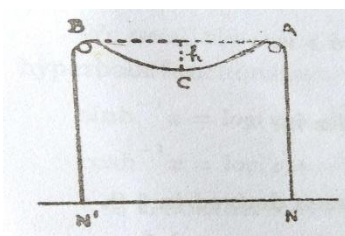
$=w^2y^2$ using equation (6) of page 377

$\therefore T = wy$ ... … … (4)

Thus the tension at any point is proportional to the height of the point above the origin. It is equal to the weight of a portion of the string whose length is equal to the height of the point above the directrix.

Important Corollary:

Suppose a long chains is thrown over two smooth pegs A and B and is in equilibrium with the portions AN and BN' hanging vertically. The potion BCA of the chain will from a catenary.



The tension of the chain is unaltered by passing overt the smooth peg A. The tension at A can be calculated by two methods.

On one side (i.e. from the catenary portion), Tension at A = $w.y$ where y is the height of A above the directrix.

On the other side, tension at A = weight of the free part AN hanging down

$= w.$ AN

$\therefore y$=AN

In other words, N is on the directrix of the catenary.

Similarly N' is on the directrix.

Hence if a long chain is thrown over two smooth pegs and is in equilibrium, the free ends must reach the directrix of the catenary formed by it.

Important Formulae:

The Cartesian coordinates of a point P on the catenary are (x, y) and its intrinsic coordinates are (s, Ψ). Hence there are four variable quantities we can have a relation connecting any two of them. There will be ${}_4C_2 = 6$ such relations, most of them having been already derived. We shall derive the remaining. It is worthwhile to collect these results for ready reference.

(i) The relation connecting x and y is

$y = c\cosh\frac{x}{c}$ … … (1)

and this is the Cartesian equation to the catenary.

(ii) The relation connecting s and Ψ is

$s = c\tan\Psi$ … … (2)

(iii) The relation connecting y and Ψ is

$y=c\sec\Psi$ … … … (3)

(iv) The relation connecting y and s is

$y^2 = c^2+s^2$ …. … … (4)

(v) The relation connecting s and x is

$s = c\sinh\frac{x}{c}$

(vi) We have $y = c\cosh\frac{x}{c}$ and $y = c\sec\Psi$,

$\therefore \sec\Psi = \cosh\frac{x}{c}$

$\therefore \frac{x}{c} = \cosh{-1}(\sec\Psi)$

$= \log(sec\Psi + \sqrt{sec^2\Psi - 1}$

$= \log(sec\Psi + \tan\Psi)$

$\therefore x = clog\ (sec\Psi + tan\Psi)$ … … (6)

This relation can also be obtained thus:

$\frac{dx}{d\Psi} = \frac{dx}{ds}.\frac{ds}{d\Psi}$

$= \cos\Psi.\frac{d}{d\Psi}(c\tan\Psi)$ since $\frac{dx}{ds} = \cos\Psi$ for any curve

$= \cos\Psi.$ Csec2Ψ – csecΨ

Integrating, $x = \int c\sec\Psi \, d\Psi + D$

$= c\log(\sec\Psi + \text{ran}\Psi) + D$

At the lowest point, $\Psi = 0$ and $x = 0$

$\therefore 0 = c\log(\sec 0 + \tan 0 + D$

i.e. $0 = D$

$\therefore x = c\log(\sec\Psi + \tan\Psi)$

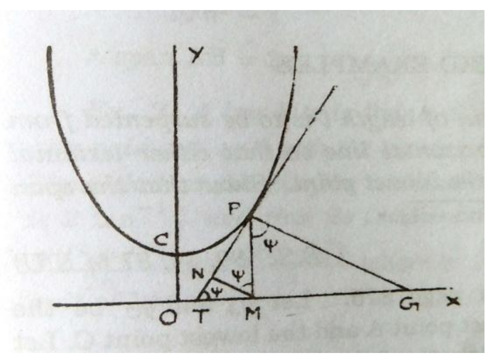
(vii) The tension at any point = wy … … (7), where y is the distance of the point from the directrix.

(viii) The tension at the lowest point = wc … … (8)

$\sinh^{-1} x = \log(x+\sqrt{x^2+1})$

$\cosh^{-1} x = \log(x+\sqrt{x^2-1})$

2.9 Geometrical Properties of the Common catenary:



Let P be any point on the catenary $y = c\cosh\frac{x}{c}$.

PT is the tangent meeting the directrix (i.e. the x axis) at T.

angle PTX = Ψ

PM (=y) is the ordinate of P and PG is the normal at P.

Draw MN $\perp$ to PT.

From ΔPMN. MN $= PM\cos\Psi$

$= y\cos\Psi$

$= c\sec\Psi \cos\Psi$

$= c =$ constant

i.e. *The length of the perpendicular from the foot of the ordinate on the tangent at any point of the catenary is constant.*

Again $\tan\Psi = \frac{PN}{MN} = \frac{PN}{C}$

$\therefore PN = {}_C\tan\Psi = S$ arc CP

$PM^2 = NM^2 + PN^2$

$\therefore y^2 = c^2 + s^2$, a relation already obtained.

If is the radius of curvature of the catenary at P,

$P = \frac{ds}{d\Psi} = \frac{d}{d\Psi}(c\tan\Psi) = c\sec^2\Psi$

Let the normal at P cut the x axis at G.

Then PG. $\cos\Psi$ = PM = y

$\therefore PG = \frac{y}{\cos\Psi} = c\sec\Psi.\ \sec\Psi = c\sec^2\Psi$

$\therefore\ \rho = PG$

Hence the radius of curvature at any point on the catenary is numerically equal to the length of the normal intercepted between the curve and the directrix, but they are drawn in opposite directions.

Problem 13

A uniform chain of length l is to be suspended from two points in the same horizontal line so that either terminal tension is n times that at the lowest point. Show that the span must be $\frac{l}{\sqrt{n^2-1}}\log(n + \sqrt{n^2-1}$

Solution:

Tension at A = wy_A

And tension at C = $w.y_C$ since $T = wy$ at any point

Now $w.y_A = n.w.y_C$

$\therefore y_A = ny_C = nc$

But $y_A = c\cosh\frac{x_A}{c} = nc$

$\therefore \cosh\frac{x_A}{c} = n$

or $\frac{x_A}{c}$ = $\cosh^{-1} n$ = $\log(n+\sqrt{n^2-1})$

$\therefore x_A$ = $c\log(n+\sqrt{n^2-1})$ … … … (1)

We have to find c.

y^2_A = $c^2+s^2_A$, s_A denoting the length of CA.

= $c^2+\frac{l^2}{4}$ (as total length = l)

i.e. n^2c^2 = $c^2+\frac{l^2}{4}$

or c^2 = $\frac{l^2}{4(n^2-1)}$

$\therefore c$ = $\frac{l^2}{2\sqrt{n^2-1}}$ … … (2)

Substituting (2) in (1),

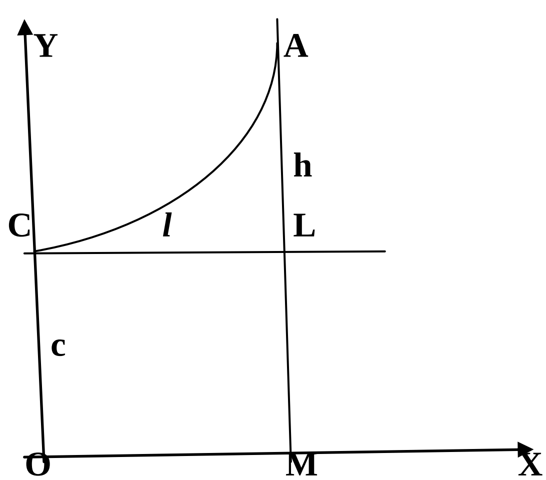
$$x_A = \frac{l^2}{2\sqrt{n^2-1}}\log(n+\sqrt{n^2-1})$$

$\therefore$ span AB = $2x_A$ = $\frac{l}{\sqrt{n^2-1)}}\log(n+\sqrt{n^2-1})$

Problem 14

A box kite is flying at a height h with a length l of wire paid out, and with the vertex of the catenary on the ground. Show that at the kite, the inclination of the wire to the ground is $2\tan^{-1}\frac{h}{l}$ and that its tensions there and at the ground are $\frac{w(l^2+h^2)}{2h}$ and $\frac{w(l^2-h^2)}{2h}$ where w is the weight of the wire per unit of length.

Solution:



C is the vertex of the catenary CA, A being the kite. The origin O is taken at a depth c below C.

Then $y_A = c + h$ and $s_A = \text{arc } CA = l$

Since $y^2 = c^2 + s^2$, we have $(c+h)^2 = c^2 + l^2$

i.e. $h^2+2ch = l^2$

or $c = \frac{l^2-h^2}{2h}$ … … … (1)

We know that $s = c \tan \Psi$ … … …. …. …. (2)

Applying (2) at the point A, we have

$l = c. \tan \Psi_A$

$\therefore$ $\tan \Psi_A = \frac{l}{c} = \frac{2hl}{l^2-h^2}$ substituting for c from (1)

$= \frac{2(\frac{h}{l})}{1-(\frac{h}{l})2}$ … … (3)

But $\tan\Psi = \frac{2\tan\frac{\Psi}{2}}{1-\tan^2\frac{\Psi}{2}}$ … … (4)

Comparing (3) and (4), we find that

$\tan\frac{\Psi}{2}$ at A $= \frac{h}{l}$

$\therefore \frac{\Psi}{2} = \tan^{-1}\frac{h}{l}$

or Ψ at A $= 2\tan^{-1}\frac{h}{l}$

The tension at A $= w.y_A$

$= w.(c + h)$

$= w\left(\frac{l^2-h^2}{2h} + h\right) = \frac{w(l^2+h^2)}{2h}$

Problem 15

A uniform chain of length l is to have its extremities fixed at two points in the same horizontal line. Show that the span must be $\frac{l}{\sqrt{8}}\log(3+\sqrt{8})$ in order that the tension at each support shall be three times that at the lowest point.

Solution:

Put n = 3 in problem number 13.

Problem 16

A uniform chain of length l is suspended from two points A, B in the same horizontal line. If the tension A is twice that at the lowest point, show that the span AB is $\frac{l}{\sqrt{3}}\log(2+\sqrt{3})$

Solution:

Put n = 2 in problem number 13.

Problem 17

A uniform chain of length $2l$ hangs between two points A and B on the same level. The tension both at A and B is five times that at the lowest point. Show that the horizontal distance between A and B is $\frac{l}{\sqrt{6}}\log(5+2\sqrt{3})$

Solution:

Put n = 5 and length = $2l$ in problem number 13.

Problem 18

If T is the tension at any point P and T_0 is the tension at the lowest point C then prove that $T^2 - T_0^2 = W^2$ where W is the weight of the arc CP of the string.

Solution:

Given T is the tension at P. Let w be the weight per unit length and y is the ordinate of P.

Then T = wy.

Also $T_0 = wc$

$$\therefore T^2 - T_0^2 = w^2y^2 - w^2c^2$$
$$= w^2(y^2 - c^2)$$
$$= w^2s^2$$
$$= W^2$$

2.10 Suspension Bridges:

In the case of a suspension bridge the main load is the weight of the roadway. We have two chains hung up so as to be parallel, their ends being firmly fixed to supports. From different points of these chains, hang supporting chains or rods which carry the roadway of the bridge. These supporting rods are spaced at equal horizontal distances from one another and so carry equal loads. The weight of the chain itself and the weights of the supporting rods may be neglected in comparison with that of the horizontal roadway. The weight supported by each of the rods may therefore be taken to be the weight of equal portions of the roadway. Hence the figure of each chain of a suspension bridge approximates very closely to that of a parabola.

UNIT III

3.1 Projectiles.

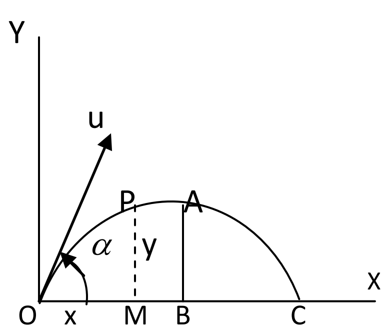
Definitions:

i. A particle projected into the air in any direction with any velocity is called a **projectile**.
ii. **The angle of projection** is the angle made by the initial velocity with the horizontal plane through the point of projection.
iii. **The velocity of projection** is the velocity with which the particle is projected.
iv. **The trajectory** is the path described by the projectile.
v. The **range on a plane** through the point of projection is the distance between the point of projection and the point where the trajectory meets that plane.
vi. **The time of flight** is the interval of time that elapses from the instant of projection till the instant when the particle again meets the horizontal plane through the point of projection.

Two fundamental principles

i. The horizontal velocity remains constant throughout the motion.
ii. The vertical component of the velocity will be subjected to retardation *g*.

3.2 Equation of the path of the projectile



Let a particle be projected from O, with initial velocity u and α be the angle of projection. Take OX and OY as x and y axes respectively. Let P (x,y) be the position of the particle in time t secs. Now u can be divided into two components as u $\cos\alpha$ in the horizontal direction and $u\sin\alpha$ in the vertical direction.

Now, horizontal velocity $u\cos\alpha$ is constant throughout the motion.

$\therefore x = (u\cos\alpha)\ t$ ………………… (1)

Vertical velocity is subjected to retardation 'g'

$\therefore y = (u\sin\alpha)\ t - \frac{1}{2}gt^2$ ………………. (2)

Eliminate 't' using (1) and (2)

$$(1) \Rightarrow t = \frac{x}{u\cos\alpha}$$

$$(2) \Rightarrow y = u\sin\alpha \frac{x}{u\cos\alpha} - \frac{1}{2}g.\left(\frac{x}{u\cos\alpha}\right)^2$$

$$y = x\tan\alpha - \frac{gx^2}{2u^2\cos^2\alpha} \qquad \text{………………. (3)}$$

$$= \frac{x\tan\alpha.2u^2\cos^2\alpha - gx^2}{2u^2\cos^2\alpha}$$

$$2u^2\cos^2\alpha.y = x.2u^2\sin\alpha\cos\alpha - gx^2$$

$$\therefore gx^2 - 2u^2\sin\alpha\cos\alpha.x = -2u^2\cos^2\alpha.y$$

$$x^2 - \frac{2u^2\sin\alpha\cos\alpha}{g}x = \frac{-2u^2\cos^2\alpha}{g}y$$

$$x^2 - \frac{2u^2\sin\alpha\cos\alpha}{g}x + \frac{u^4\sin^2\alpha\cos^2\alpha}{g^2} = \frac{u^4\sin^2\alpha\cos^2\alpha}{g^2} - \frac{2u^2\cos^2\alpha}{g}.y$$

ie) $\left(x - \frac{u^2\sin\alpha\cos\alpha}{g}\right)^2 = -\frac{2u^2\cos^2\alpha}{g}\left(y - \frac{u^2\sin^2\alpha}{2g}\right)$ ……… (4)

Shifting the origin to $\left(\frac{u^2\sin\alpha\cos\alpha}{g}, \frac{u^2\sin^2\alpha}{2g}\right)$

$$X^2 = -\frac{2u^2\cos^2\alpha}{g}.Y \qquad \text{……………….. (5)}$$

(5) is the equation of a parabola of the form $X^2 = -4aY$,

whose latus-rectum is $\dfrac{2u^2\cos^2\alpha}{g} = \dfrac{2}{g}(u\cos\alpha)^2$

$= \dfrac{2}{g}(horizontal\ \ velocity)^2$

Vertex is $\left(\dfrac{u^2\sin\alpha.\cos\alpha}{g}, \dfrac{u^2\sin^2\alpha}{2g}\right)$

3.3 Characteristics of the motion of the projectile

1. Greatest height attained by a projectile.

2. Time taken to reach the greatest height.

3. Time of flight.

4. The range on the horizontal plane through the point of projection.

Derive formula for the characteristics

3.3.1 Greatest height h

When the particle reaches the highest point at A, its direction is horizontal.

$\therefore$ At A, vertical velocity = 0

Let AB = h.

Consider the vertical motion and using the formula "$v^2 = u^2 + 2aS$"

$$O = (u\sin\alpha)^2 - 2g.h \qquad \therefore h = \frac{u^2\sin^2\alpha}{2g}$$

- ❖ Highest point of the path is the vertex of the parabola.

3.3.2 Time taken to reach the greatest height T

Let T be the time taken to travel from O to reach the greatest height at A.

At A final vertical velocity is zero

At O initial vertical velocity is $u\sin\alpha$

Using the formula "$v = u + at$"

$$O = u\sin\alpha - gT \ \therefore \boxed{T = \frac{u\sin\alpha}{g}}$$

3.3.3 Time of flight t

Let t be the time taken to travel from O to C along its path. At C, vertical distance traveled is zero. Consider the vertical motion and by the formula $S = ut + \frac{1}{2}at^2$,

$$O = u\sin\alpha \ .t - \frac{1}{2}gt^2$$

ie) $t\left(u\sin\alpha - \frac{1}{2}gt\right) = 0$

$\therefore t = 0$ or $u\sin\alpha - \frac{1}{2}gt = 0$

ie) $t = 0$ or $t = \frac{2u\sin\alpha}{g} = 2\left(\frac{u\sin\alpha}{g}\right) = 2T$

t = 0 gives the time of projection.

$$\boxed{\therefore \text{ Time of flight } t = \frac{2u\sin\alpha}{g}}$$

- Time of flight = 2 x time taken to reach the greatest height.

3.3.4 The range on the horizontal plane through the point of projection R

Range R = OC = horizontal distance traveled during the time of flight.

= horizontal velocity x time of flight

$$= u\cos\alpha \times \frac{2u\sin\alpha}{g} = \frac{2u^2\sin\alpha\cos\alpha}{g} = \frac{u^2\sin 2\alpha}{g}$$

- Horizontal range R = $\frac{2(u\cos\alpha)(u\sin\alpha)}{g} = \frac{2UV}{g}$

Where U – initial horizontal velocity, V – initial vertical velocity.

Problem 1

A body is projected with a velocity of 98 metres per sec. in a direction making an angle $\tan^{-1} 3$ with the horizon; show that it rises to a vertical height of 441 metres and that its time of flight is about 19 sec. Find also horizontal range through the point of projection (g=9.8 metres / sec^2)

Solution:

Given u = 98; α = tan $^{-1}$3 i.e tan α = 3

$$\therefore \sin\alpha = \frac{\sin\alpha}{\cos\alpha}\cdot\cos\alpha = \frac{\tan\alpha}{\sec\alpha} = \frac{\tan\alpha}{\sqrt{1+\tan^2\alpha}} = \frac{3}{\sqrt{10}}$$

$$\cos\alpha = \frac{\sin\alpha}{\tan\alpha} = \frac{1}{\sqrt{10}}$$

Greatest height = $\dfrac{u^2\sin^2\alpha}{2g} = \dfrac{98\times 98\times 9}{10\times 2\times 9.8}$ = 441 metres

Time of flight = $\dfrac{2u\sin\alpha}{g} = \dfrac{2\times 98\times 3}{\sqrt{10}\times 9.8} = 6\sqrt{10}$

= $6\times 3.162 = 18.972 = 19$ secs. nearly

Horizontal range = $\dfrac{2u^2\sin\alpha\cos\alpha}{g}$

= $\dfrac{2\times 98\times 98}{9.8}\times\dfrac{3}{\sqrt{10}}\times\dfrac{1}{\sqrt{10}}$ = 588 metres

Problem 2

If the greatest height attained by the particle is a quarter of its range on the horizontal plane through the point of projection, find the angle of projection

Solution

Let u be the initial velocity and α the angle of projection

Greatest height = $\dfrac{u^2\sin^2\alpha}{2g}$

Horizontal range = $\dfrac{2u^2 \sin\alpha \cos\alpha}{g}$

Given $\dfrac{u^2 \sin^2 \alpha}{2g} = \dfrac{1}{4} \times \dfrac{2u^2 \sin\alpha \cos\alpha}{g}$

i.e $\dfrac{u^2 \sin^2 \alpha}{2g} = \dfrac{u^2 \sin\alpha \cos\alpha}{2g}$

i.e $\sin\alpha = \cos\alpha \Rightarrow \tan\alpha = 1 \quad \therefore \alpha = 45^0$

Problem 3

A particle is projected so as to graze the tops of two parallel walls, the first of height 'a' at a distance b from the point of projection and the second of height b at a distant 'a' from the point of projection. If the path of particle lies in a plane perpendicular to both the walls, find the range on the horizontal plane and show that the angle of projection exceeds $\tan^{-1}3$.

Solution:

Let u be the initial velocity, α be the angle of projection.

Equation to the path is $y = x\tan\alpha - \dfrac{gx^2}{2u^2\cos^2\alpha}$

i.e $y = xt - \dfrac{gx^2}{2u^2}\left(1+t^2\right)$ where $t = \tan\alpha$ ……. (1)

The tops of the two walls are (b, a) and (a, b) lie on (1)

$\therefore\ a = bt - \dfrac{gb^2}{2u^2}\left(1+t^2\right)$ ……… (2)

b = $at - \dfrac{ga^2}{2u^2}\left(1+t^2\right)$ ………. (3)

From (2), $a - bt = -\dfrac{gb^2}{2u^2}\left(1+t^2\right)$ ………. (4)

From (3), $b - at = -\dfrac{ga^2}{2u^2}\left(1+t^2\right)$ ………. (5)

Dividing (4) by (5), $\frac{a-bt}{b-at}=\frac{b^2}{a^2}$

i.e $b^3 - ab^2\, t = a^3 - a^2bt \qquad \Rightarrow \qquad t\,(a^2b - ab^2) = a^3 - b^3$

$$\therefore t=\frac{a^3-b^3}{a^2b-ab^2}=\frac{(a-b)\left(a^2+ab+b^2\right)}{ab(a-b)}=\frac{a^2+ab+b^2}{ab}$$

$$\therefore \tan\alpha=\frac{a^2+ab+b^2}{ab}=\frac{\left(a^2-2ab+b^2\right)+3ab}{ab}=\frac{(a-b)^2}{ab}+3 \text{.....(6)}$$

$$(6)\Rightarrow \tan\alpha>3 \;\; or \;\; \alpha>\tan^{-1}3$$

From (4), $\frac{g\left(1+t^2\right)}{2u^2}=\frac{a-bt}{-b^2}=\frac{bt-a}{b^2}$

$$=\frac{\frac{b\left(a^2+ab+b^2\right)}{ab}-a}{b^2}=\frac{a^2+ab+b^2-a^2}{ab^2}$$

$$=\frac{b(a+b)}{ab^2}=\frac{a+b}{ab} \qquad \text{........... (7)}$$

Horizontal range = $\frac{u^2\sin 2\alpha}{g} = \frac{2u^2t}{g\left(1+t^2\right)} \quad \because \sin 2\alpha=\frac{2\tan\alpha}{1+\tan^2\alpha}$

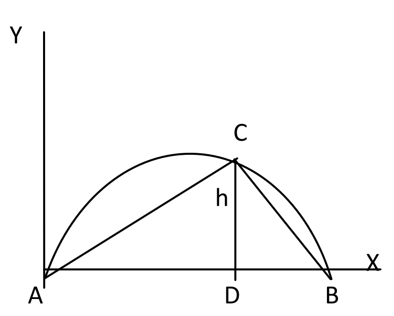
$$= t.\frac{ab}{a+b} \text{ from (7)}$$

$$= \frac{\left(a^2+ab+b^2\right)}{ab}\cdot\frac{ab}{a+b} = \frac{a^2+ab+b^2}{a+b}$$

Problem 4

A particle is thrown over a triangle from one end of a horizontal base and grazing the vertex falls on the other end of the base. If A, B are the base angles, and α the angle of projection, show that $\tan\alpha$ = tan A + tan B

Solution:



Let u be the velocity of projection and α the angle of projection and let t secs be the time taken from A to C. Draw CD $\perp$ AB and let CD = h.

Consider the vertical motion, h = vertical distance described in time t

$$= u \sin \alpha \cdot t - \frac{1}{2} g t^2$$

AD = horizontal distance described in time t = u cos $\alpha \cdot$t

From ΔCAD, $\tan A = \dfrac{CD}{AD} = \dfrac{h}{AD} = \dfrac{u \sin \alpha \cdot t - \frac{1}{2} g t^2}{u \cos \alpha \cdot t}$

$$= \tan \alpha - \frac{gt}{2u \cos \alpha} \qquad \ldots\ldots (1)$$

AB = horizontal range = $\dfrac{2u^2 \sin \alpha \cos \alpha}{g}$

$\therefore$ DB = AB – AD = $\dfrac{2u^2 \sin \alpha \cos \alpha}{g} - u \cos \alpha \cdot t$

From ΔCDB, $\tan B = \dfrac{CD}{DB} = \dfrac{h}{\left(\dfrac{2u^2 \sin \alpha \cos \alpha}{g} - u \cos \alpha \cdot t\right)}$

$$= \frac{u \sin \alpha . t - \frac{1}{2} g t^2}{\left(\dfrac{2u^2 \sin \alpha \cos \alpha}{g} - u \cos \alpha . t\right)}$$

$$= \frac{gt\left(u \sin \alpha - \frac{1}{2} g t\right)}{u \cos \alpha (2u \sin \alpha - gt)}$$

$$= \frac{gt(2u \sin \alpha - gt)}{2u \cos \alpha (2u \sin \alpha - gt)} = \frac{gt}{2u \cos \alpha} \ldots\ldots\ldots\ldots (2)$$

(1) + (2) $\Rightarrow$ tanA + tanB = tanα

Problem 5

Show that the greatest height which a particle with initial velocity v can reach on a vertical wall at a distance ‘a’ from the point of projection is $\frac{v^2}{2g}-\frac{ga^2}{2v^2}$ Prove also that the greatest height above the point of projection attained by the particle in its fight is $v^6/2g\left(v^4+g^2a^2\right)$

Solution:

Equation to the path is $y = x\tan\alpha - \frac{gx^2}{2v^2\cos^2\alpha}$ …….. (1)

Put x = a in (1), $y = a\tan\alpha - \frac{ga^2}{2v^2\cos^2\alpha}$

$y = at - \frac{ga^2}{2v^2}\left(1+t^2\right)$ where t = tan α …….. (2)

y is a function of t. $\therefore$ y is maximum when $\frac{dy}{dt}=0$ and $\frac{d^2y}{dt^2}$ is negative.

Differentiating (2) with respect to t,

$$\frac{dy}{dt} = a - \frac{ga^2}{2v^2}\cdot 2t = a - \frac{ga^2t}{v^2}$$

$$\frac{d^2y}{dt^2} = -\frac{ga^2}{v^2} = \text{negative}$$

So y is maximum when $a - \frac{ga^2t}{v^2} = 0$ or $t = \frac{v^2}{ga}$ ……(3)

Put $t = \frac{v^2}{ga}$ in (2)

Max value of $y = a\cdot\frac{v^2}{ga} - \frac{ga^2}{2v^2}\left(1+\frac{v^4}{g^2a^2}\right)$

$$= \frac{v^2}{g} - \frac{ga^2}{2v^2} - \frac{v^2}{2g} = \frac{v^2}{2g} - \frac{ga^2}{2v^2}$$

Greatest height during the flight

$$= \frac{v^2\sin^2\alpha}{2g} = \frac{v^2}{2g}\cdot\frac{1}{\operatorname{cosec}^2\alpha} = \frac{v^2}{2g\left(1+\cot^2\alpha\right)}$$

$$= \frac{v^2}{2g\left(1+\frac{g^2a^2}{v^4}\right)} \text{ from (3)}$$

$$= \frac{v^6}{2g\left(v^4+g^2a^2\right)}$$

Problem 6

a. A projectile is thrown with a velocity of 20 m/sec. at an elevation 30^0. Find the greatest height attained and the horizontal range.

b. A particle is projected with a velocity of 9.6 metres at an angle of 30^0. Find

i. The time of flight

ii. the greatest height of the particle.

Solution:

Given u = 20m/sec; $\alpha = 30^0$

Greatest height = $\frac{u^2 \sin^2 \alpha}{2g} = \frac{20^2\left(\sin 30^0\right)^2}{2\times 9.8} = 5.1m$

Horizontal range = $\frac{u^2 \sin 2\alpha}{g} = \frac{20^2 \cdot \sin 60^0}{9.8} = 35.35m$

Problem 7

(a) A particle is projected under gravity in a vertical plane with a velocity u at an angle α to the horizontal. If the range on the horizontal be R and the greatest height attained by h, show that $\frac{u^2}{2g} = h + \frac{R^2}{16h}$ and $\tan\alpha = \frac{4h}{R}$.

(b) A particle is projected so that on its upward path, it passes through a point x feet horizontally and y feet vertically from the point of projection. Show that, if R be the horizontal range, the angle of projection is $\tan^{-1}\left(\frac{y}{x}\cdot\frac{r}{R-x}\right)$.

Solution:

a) $$h+\frac{R^2}{16h}=\frac{u^2\sin^2\alpha}{2g}+\frac{\left(\frac{2u^2\sin\alpha\cos\alpha}{g}\right)^2}{16\left(\frac{u^2\sin^2\alpha}{2g}\right)}$$

$$=\frac{u^2\sin^2\alpha}{2g}+\frac{u^2\cos^2\alpha}{2g}=\frac{u^2}{2g}$$

b) Equation of the path is, $y=x\tan\alpha-\frac{gx^2}{2u^2\cos^2\alpha}$

$$\therefore x\tan\alpha=y+\frac{gx^2}{2u^2\cos^2\alpha}$$

$$\therefore \tan\alpha=\frac{y}{x}+\frac{gx}{2u^2\cos^2\alpha} \qquad \text{...........} \ (1)$$

We have $R=\frac{2u^2\sin\alpha\cdot\cos\alpha}{g}\Rightarrow g=\frac{2u^2\sin\alpha\cos\alpha}{R}$

$$\therefore (1)\Rightarrow \tan\alpha=\frac{y}{x}+\frac{x}{2u^2\cos^2\alpha}\times\frac{2u^2\cdot\sin\alpha\cos\alpha}{R}=\frac{y}{x}+\frac{x\tan\alpha}{R}$$

$$\therefore \tan\alpha\left(1-\frac{x}{R}\right)=\frac{y}{x}$$

ie $\tan\alpha\left(\frac{R-x}{R}\right)=\frac{y}{x}$ or $\tan\alpha=\frac{y}{x}.\frac{R}{R-x}$

$$\therefore \alpha=\tan^{-1}\left(\frac{y}{x}\cdot\frac{R}{R-x}\right)$$

Problem 8

If the time of flight of a shot is T seconds over a range of x metres, show that the elevation is $\tan^{-1}\left(\frac{gT^2}{2x}\right)$ and determine the maximum height and the velocity of projection.

Solution:

Given, horizontal range R = x metres

Time of flight $T = \dfrac{2u\sin\alpha}{g}$ ………. (1)

where α -is the angle of projection

$$\therefore x = \frac{2u^2 \sin\alpha\cos\alpha}{g}$$

$\therefore$ (1) $\Rightarrow$ gT = 2u sinα . $\Rightarrow$ $\boxed{u = \dfrac{gT}{2\sin\alpha}}$

$$\therefore x = \frac{2\cdot g^2T^2\cdot\sin\alpha\cos\alpha}{4\sin^2\alpha\cdot g} = \frac{1}{2}gT^2\cdot\cot\alpha$$

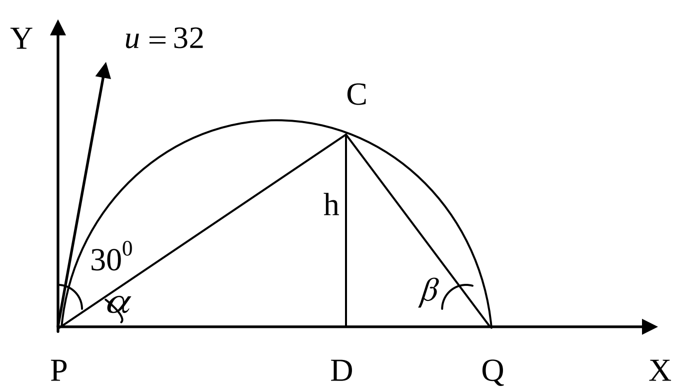
$$\therefore \tan\alpha = \frac{gT^2}{2x} \Rightarrow \boxed{\alpha = \tan^{-1}\left(\frac{gT^2}{2x}\right)}$$

Maximum height = $\dfrac{u^2\sin^2\alpha}{2g} = \dfrac{g^2T^2}{4\sin^2\alpha}\cdot\dfrac{\sin^2\alpha}{2g} = \dfrac{gT^2}{8}$

Problem 9

A particle is projected from a point P with a velocity of 32m per second at an angle of 30^0 with the horizontal. If PQ be its horizontal range and if the angles of elevation from P and Q at any instant of its flight be α and β respectively, show that $\tan\alpha + \tan\beta = \dfrac{1}{\sqrt{3}}$

Solution:



Given, initial velocity u = 32 m/sec, 30^0 is the angle of projection. P-be the point of projection. 't' – be the time taken from P to C.

Let CD = h = $u\sin\alpha.t - \frac{1}{2}gt^2$

$h = (32.\sin 30^0)\ t - \frac{1}{2}gt^2 =$ vertical distance described in t secs

$= 16t - \frac{1}{2}gt^2$

PD = horizontal distance described in t secs = $u\cos\alpha.t$

$= \left(32\cos 30^0\right) t \quad = 32\cdot\frac{\sqrt{3}}{2}t \quad = 16\sqrt{3}t$.

From Δ PCD , $\tan\alpha = \frac{h}{PD} = \frac{h}{16\sqrt{3}t}$ …….. (1)

From Δ QCD, $\tan\beta = \frac{h}{DQ} = \frac{h}{PQ - PD}$, PQ = range

ie $$\tan\beta = \frac{h}{\left(\frac{2(32)^2\sin 30^0\cdot\cos 30^0}{g}\right) - 16\sqrt{3}t}$$

$$= \frac{hg}{512\sqrt{3} - 16\sqrt{3}gt} \quad \text{………(2)}$$

$$\therefore (1) + (2) \Rightarrow \tan\alpha + \tan\beta = \frac{h}{16\sqrt{3}}\left[\frac{1}{t} + \frac{g}{32 - gt}\right]$$

$$= \frac{\left(16t - \frac{1}{2}gt^2\right)}{16\sqrt{3}}\left[\frac{32 - gt + gt}{t(32 - gt)}\right]$$

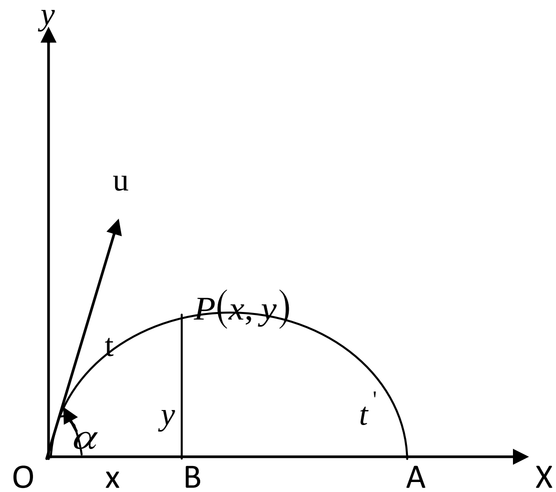
$$= \frac{t\left(32 - gt\right)}{32\sqrt{3}} \times \frac{32}{t(32 - gt)} = \frac{1}{\sqrt{3}}$$

$$\therefore \quad \boxed{\tan\alpha + \tan\beta = \frac{1}{\sqrt{3}}}$$

Problem 10

A particle is projected and after time t reaches a point P. If t is the lime it takes to move from P to the horizontal plane through the point of projection, prove that the height of P above the plane is $\frac{1}{2}gt\ t^{'}$

Solution:



Let u be the velocity of projection, α be the angle of projection, P be the position of the particle after t secs. Let $t^{'}$ be the time taken to travel from P to A

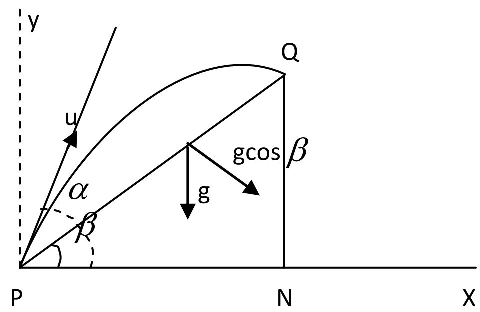
$\therefore$ We have $t+t^{'}$ = time of flight = $\frac{2u\sin\alpha}{g}$ $\therefore u\sin\alpha = \frac{g(t+t^{'})}{2}$

Now, y = vertical distance described in t secs $= (u\sin\alpha)\ t - \frac{1}{2}gt^2$

$$= \frac{g(t+t^{'})\ t}{2} - \frac{1}{2}gt^2 \qquad = \frac{gtt^{'}}{2}$$

$\therefore$ Height of P above the plane = $= \frac{gtt^{'}}{2}$

3.4 Range on an inclined Plane:



Let P be the point of projection on a plane of inclination β, u be the velocity of projection at an angle α with the horizontal. The particle strikes the inclined plane at Q. Then PQ = r is the range on the inclined plane. Take PX and PY as x and y axes.

Draw $QN \perp PX$.

From $\Delta PQN,\ PN = r\cos\beta,\ QN = r\sin\beta$

$Q(r\cos\beta, r\sin\beta)$ lies on the path. $y = x\tan\alpha - \dfrac{gx^2}{2u^2\cos^2\alpha}$

$$\therefore r\sin\beta = r\cos\beta.\tan\alpha - \frac{g(r\cos\beta)^2}{2u^2\cos^2\alpha}$$

Dividing by r we get $\dfrac{gr\cos^2\beta}{2u^2\cos^2\alpha} = \cos\beta.\dfrac{\sin\alpha}{\cos\alpha} - \sin\beta$

$$\therefore r = \frac{2u^2\cos^2\alpha}{g\cos^2\beta}\left[\frac{\sin\alpha\cos\beta - \cos\alpha\sin\beta}{\cos\alpha}\right]$$

$$\boxed{r = \frac{2u^2\cos\alpha}{g\cos^2\beta}\sin(\alpha-\beta)}$$

3.5 Maximum range on the inclined plane, given u the velocity of projection and β the inclination of the plane:

Range r on the inclined plane is

$$r = \frac{2u^2 \cos\alpha \sin(\alpha-\beta)}{g\cos^2\beta} = \frac{u^2}{g\cos^2\beta}\left[\sin(2\alpha-\beta) - \sin\beta\right] \quad \text{..... (1)}$$

Now u and β are given, g constant.

So r is maximum when $\left[\sin(2\alpha-\beta) - \sin\beta\right]$ is maximum.

i.e. when sin ($2\alpha-\beta$) is maximum.

i.e.when. $2\alpha - \beta = \dfrac{\pi}{2}$

$$\therefore \quad \boxed{\alpha = \frac{\pi}{4} + \frac{\beta}{2}} \quad \text{for maximum range.}$$

From (1), maximum range on the inclined plane

$$= \frac{u^2}{g\cos^2\beta}(1-\sin\beta) = \frac{u^2}{g(1+\sin\beta)}$$

3.5.1 Time of flight T (up an inclined plane):

From the figure in 6.11, the time taken to travel from P to Q is the time of flight. Consider the motion perpendicular to the inclined plane. At the end of time T, the distance travelled perpendicular to the inclined plane S = 0, component of g perpendicular to the inclined plane is $g\cos\beta$, initial velocity perpendicular to the inclined plane is $u\sin(\alpha-\beta)$.

$$0 = u\sin(\alpha-\beta)T - \frac{1}{2}g\cos\beta.T^2 \qquad \text{using } "S = ut + \frac{1}{2}at^2"$$

$$\boxed{\therefore T = \frac{2u\sin(\alpha-\beta)}{g\cos\beta}}$$

3.5.2 Greatest distance S of the projectile from the inclined plane and show that it is attained in half the total time of flight:

Consider the motion perpendicular to the inclined plane. The initial velocity perpendicular to the plane is u sin (α - β) and this is subjected to an acceleration gcosβ in the same direction but acting downwards. Let S be the greatest distance travelled by the particle perpendicular to the inclined plane. At the greatest distance the velocity becomes parallel to the inclined plane and hence the velocity perpendicular to the plane is zero.

Using the formula "$v^2 = u^2 + 2as$"

$$0 = [u\sin(\alpha - \beta)]^2 - 2g\cos\beta . S$$

$$\boxed{S = \frac{u^2 . \sin^2(\alpha - \beta)}{2.g\cos\beta}}$$

3.5.3 Time taken to reach the greatest distance t :

When the particle is at the greatest distance from the inclined plane, its velocity becomes parallel to the inclined plane and the velocity perpendicular to the inclined plane is zero. So, if t is the time taken to reach the greatest distance, using the formula

"$v = u + at$"

$$\therefore 0 = [u\sin(\alpha - \beta)] - g\cos\beta \cdot t$$

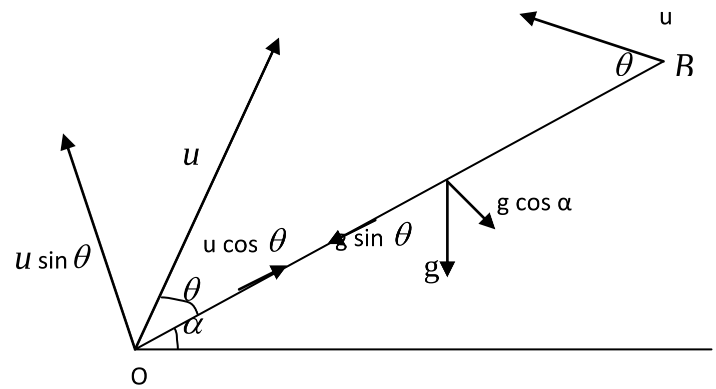
i.e. $\boxed{t = \dfrac{u\sin(\alpha - \beta)}{g\cos\beta}}$

Note : Time of flight T = $\dfrac{2u\sin(\alpha - \beta)}{g\cos\beta}$ = 2.t = 2 $\times$ time taken to reach the greatest distance.

Problem 11

Show that, for a given velocity of projection the maximum range down an inclined plane of inclination α bears to the maximum range up the inclined plane the ratio $\dfrac{1+\sin\alpha}{1-\sin\alpha}$

Solution



Let u be the given velocity of projection and θ the inclination of the direction of projection with the plane. u has two components u$\cos\theta$ along the upward inclined plane and u$\sin\theta$ perpendicular to the inclined plane. g has two components, g sin α along the downward inclined plane and g$\cos\alpha$ perpendicular to the inclined plane and downwards.

Consider the motion perpendicular to the inclined plane. Let T be the time of flight. Distance travelled perpendicular to the inclined plane in time T = 0

$$\therefore 0 = u\sin\theta\cdot T - \frac{1}{2}g\cos\alpha\cdot T^2 \qquad \left(\because S = ut + \frac{1}{2}at^2\right)$$

i.e. $T = \dfrac{2u\sin\theta}{g\cos\alpha}$

Range up the plane = R_1

R_1= distance travelled along the plane in time T

$$= u\cos\theta\cdot T - \frac{1}{2}g\sin\alpha\cdot T^2$$

$$= u\cos\theta\cdot\frac{2u\sin\theta}{g\cos\alpha} - \frac{1}{2}g\sin\alpha\cdot\frac{4u^2\sin^2\theta}{g^2\cos^2\alpha}$$

$$= \frac{2u^2\sin\theta\cos\theta}{g\cos\alpha} - \frac{2u^2\sin\alpha\sin^2\theta}{g\cos^2\alpha}$$

$$= \frac{2u^2\sin\theta}{g\cos^2\alpha}(\cos\alpha\cos\theta - \sin\alpha\sin\theta)$$

$$= \frac{2u^2 \sin\theta}{g\cos^2\alpha}\cos(\theta+\alpha) = \frac{u^2}{g\cos^2\alpha}\cdot 2\cos(\theta+\alpha)\sin\theta$$

$$= \frac{u^2}{g\cos^2\alpha}\left[\sin(2\theta+\alpha)-\sin\alpha\right]$$

R_1 is maximum, when $\sin(2\theta+\alpha)=1$

$\therefore$ Maximum range up the plane

$$= \frac{u^2}{g\cos^2\alpha}(1-\sin\alpha) = \frac{u^2}{g(1+\sin\alpha)} \quad \ldots\ldots\ldots\ldots (1)$$

When the particle is projected down the plane from B at the same angle θ to the plane, the time of flight T has the same value $\frac{2u\sin\theta}{g\cos\alpha}$. The component of the initial velocity along the inclined plane is u $\cos\theta$ downwards and the component of acceleration g sin α is also downwards.

Range down the plane = R_2

R_2 = distance travelled along the plane in time T

$$= u\cos\theta\cdot T + \frac{1}{2}g\sin\alpha\cdot T^2$$

$$= \frac{2u^2\sin\theta}{g\cos^2\alpha}(\cos\alpha\cos\theta+\sin\alpha\sin\theta)$$

$$= \frac{2u^2\sin\theta}{g\cos^2\alpha}\cos(\theta-\alpha) = \frac{u^2}{g\cos^2\alpha}\left[\sin(2\theta-\alpha)+\sin\alpha\right]$$

R_2 is maximum, when $\sin(2\theta-\alpha) = 1$.

Maximum range down the plane

$$= \frac{u^2}{g\cos^2\alpha}(1+\sin\alpha) = \frac{u^2}{g(1-\sin\alpha)} \quad \ldots\ldots\ldots\ldots.. (2)$$

$$\therefore \frac{Max\cdot range\ down\ the\ plane}{Max\cdot range\ up\ the\ plane} = \frac{u^2}{g(1-\sin\alpha)}\cdot\frac{g(1+\sin\alpha)}{u^2} = \frac{1+\sin\alpha}{1-\sin\alpha}$$

Problem 12

A particle is projected at an angle α with a velocity u and it strikes up an inclined plane of inclination β at right angles to the plane. Prove that (i) cot β = 2tan(α - β) (ii) cot β = tan $\alpha - 2\tan\beta$. If the plane is struck horizontally, show that tan $\alpha = 2\tan\beta$.

Solution:

The initial velocity and acceleration are split into components along the plane and perpendicular to the plane.

The time of flight is $T = \dfrac{2u\sin(\alpha-\beta)}{g\cos\beta}$ ….. (1)

Since the particle strikes the inclined plane normally, its velocity parallel to the inclined plane at the end of time T is = 0.

i.e. 0 = u cos (α - β) – g sin $\beta \cdot T$

$$T = \frac{u\cos(\alpha-\beta)}{g\sin\beta} \quad \text{….. (2)}$$

$$\frac{2u\sin(\alpha-\beta)}{g\cos\beta} = \frac{u\cos(\alpha-\beta)}{g\sin\beta} \text{ from (1) and (2)}$$

i.e. cot β = 2 tan ($\alpha - \beta$) …… (i)

i.e. $\cot\beta = \dfrac{2(\tan\alpha - \tan\beta)}{1+\tan\alpha\tan\beta}$, Simplifying we get

cot β + tan α = 2 tan $\alpha - 2$ tan β

cot β = tan $\alpha - 2$ tan β ….. (ii)

If the plane is struck horizontally, the vertical velocity of the projectile at the end of time T = 0. Initial vertical velocity = u sin α, and acceleration in this direction = g (downwards).

Vertical velocity in time T = u sin $\alpha -$ gT

$\therefore$ u sin $\alpha - \text{g}T = 0$ or T = $\dfrac{u\sin\alpha}{g}$ ….. (3)

$$\frac{2u\sin(\alpha-\beta)}{g\cos\beta} = \frac{u\sin\alpha}{g}$$ from (1) and (3)

Simplifying we get

2 sin ($\alpha - \beta$) = sin α cos β

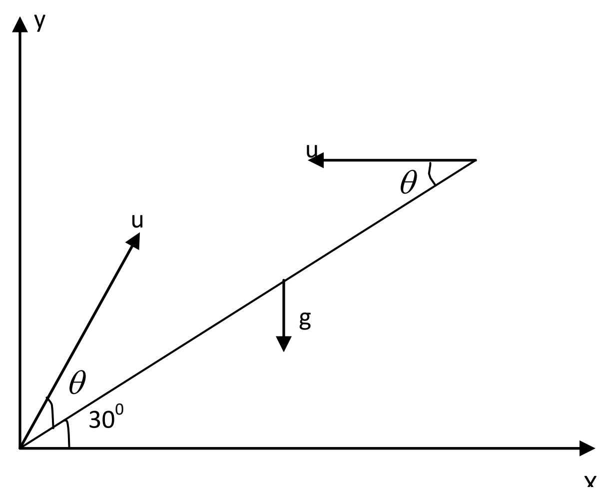
2(sin α cos β – cos α sin β) = sin α cos β.

sin α cos β = 2cos α sin β or tan α = 2tan β

Problem 13

The greatest range with a given velocity of projection on a horizontal plane is 3000 metres. Find the greatest ranges up and down a plane inclined at 30^0 to the horizon.

Solution:



Let u be the velocity of projection, θ be the inclination of direction of projection with the plane. Given $\dfrac{u^2}{g} = 3000\,m \Rightarrow u^2 = 3000 \times g$

At the end of time t, distance travelled perpendicular to the inclined plane is zero.

$$\therefore 0 = u\sin\theta \cdot T - \frac{1}{2} g\cos 30^0 \cdot T^2$$

$$0 = u\sin\theta \cdot T - \frac{1}{2} g \cdot \frac{\sqrt{3}}{2} \cdot T^2$$

$$\boxed{\therefore T = \frac{4u\sin\theta}{g\sqrt{3}}}$$

Range up the inclined plane, S = $u\cos\theta \cdot T - \frac{1}{2} g \cdot \sin 30^0 \cdot T^2$

$$= u\cos\theta \cdot \frac{4u\sin\theta}{g\sqrt{3}} - \frac{1}{4} \cdot g \cdot \times \frac{16u^2\sin^2\theta}{3g^2}$$

$$= \frac{4u^2\sin\theta\cos\theta}{g\sqrt{3}} - \frac{4u^2\sin^2\theta}{3g}$$

$$S = \frac{4u^2\sin\theta}{3g}\left[\sqrt{3}\cos\theta - \sin\theta\right]$$

Max. range is got when $\sin(2\theta + 30^0) = 1$

i.e. $2\theta + 30^0 = 90^0 \therefore \theta = 30^0$

Max. range up the inclined plane

$$= S_{\max} = \frac{4u^2\sin 30^0}{3g}\left[\sqrt{3}\cos 30^0 - \sin 30^0\right]$$

$$= \frac{4u^2 \times \frac{1}{2}}{3g}\left[\sqrt{3} \times \frac{\sqrt{3}}{2} - \frac{1}{2}\right] = \frac{2}{3} \times 3000 \quad S_{\max} = 2000m$$

$\therefore$ Range down the inclined plane = $\frac{u^2}{g\cos^2\alpha}\left[\sin(2\theta - \alpha) + \sin\alpha\right]$

Max. range down the inclined plane

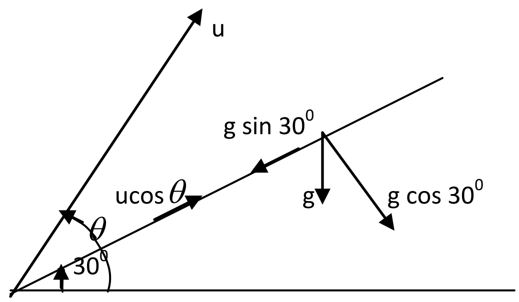
$$= \frac{u^2}{g \cdot \cos^2 30^0}\left[1 + \sin 30^0\right] = \frac{4u^2}{3g}\left[1 + \tfrac{1}{2}\right]$$

$$= \frac{2u^2}{g} = 2 \times 3000 = 6000m$$

Problem 14

An inclined plane is inclined at an angle of 30^0 to the horizon. Show that, for a given velocity of projection, the maximum range up the plane is 1/3 of the maximum range down the plane.

Solution:



Max. range up the plane = $\dfrac{u^2}{g\cdot\cos^2 30^0}\left[1-\sin 30^0\right] = \dfrac{2u^2}{3g}$

Max. range down the plane = $\dfrac{u^2}{g\cdot\cos^2 30^0}\left[1+\sin 30^0\right]$

$$= \frac{4u^2}{3g}\times\frac{3}{2}=\frac{2u^2}{g}$$

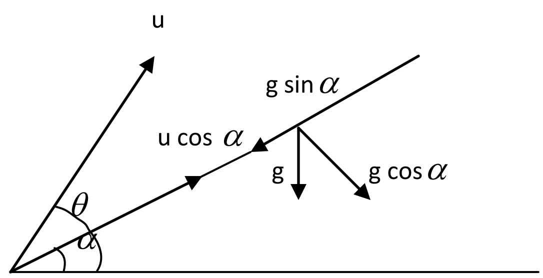
Max. range up the plane = $\dfrac{1}{3}\times\dfrac{2u^2}{g}$

$$= \frac{1}{3}\times \text{max}\cdot \textit{range down the plane}$$

Problem 15

If the greatest range down an inclined plane is three times its greatest range up the plane then show that the plane is inclined at 30^0 to the horizon..

Solution



Greatest range down the inclined plane R_1

$$R_1 = \frac{u^2}{g\cos^2\alpha}[1+\sin\alpha]$$

Greatest range down the inclined plane R_2

$$R_2 = \frac{u^2}{g\cos^2\alpha}[1-\sin\alpha]$$

Given, $R_1 = 3R_2$

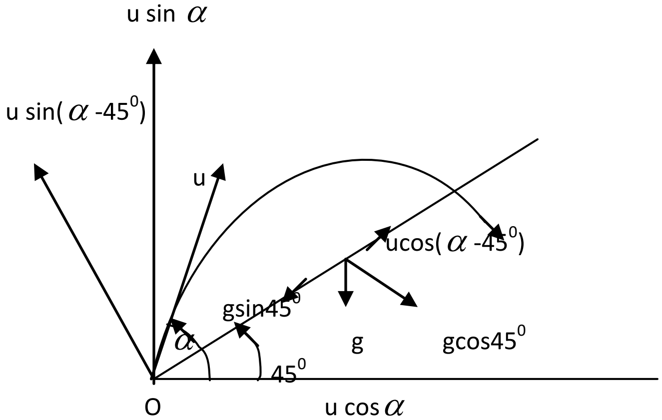
$$\text{i.e. } \frac{u^2}{g\cos^2\alpha}[1+\sin\alpha] = 3\cdot\frac{u^2}{g\cos^2\alpha}[1-\sin\alpha]$$

$$\sin\alpha = \frac{1}{2} \quad \therefore \alpha = 30^0$$

Problem 16

A particle is projected in a vertical plane at an angle α to the horizontal from the foot of a plane whose inclination to the horizon is 45^0. Show that the particle will strike the plane at right angles if tan α =3.

Solution:



When the particle strikes the plane at right angles, velocity parallel to the plane is zero.

$$\therefore O = u\cos(\alpha - 45^0) - g \cdot \sin 45^0 \cdot T$$

$$\therefore T = \frac{u\cos(\alpha - 45^0)}{g\sin 45^0} = \frac{u\cos(\alpha - 45^0)}{g \cdot \frac{1}{\sqrt{2}}} \qquad \ldots\ldots. (1)$$

Also, time of flight, $T = \dfrac{2u \cdot \sin\left(\alpha - 45^0\right)}{g \cdot \cos 45^0}$ ……. (2)

$$(1)\ \&\ (2) \Rightarrow \frac{u\cos\left(\alpha - 45^0\right)}{g \cdot \frac{1}{\sqrt{2}}} = \frac{2u \cdot \sin\left(\alpha - 45^0\right)}{g. \frac{1}{\sqrt{2}}}$$

$$\Rightarrow \cos\left(\alpha - 45^0\right) = 2 \cdot \sin(\alpha - 45^0) \ \Rightarrow 2 \cdot \tan\left(\alpha - 45^0\right) = 1$$

$$\Rightarrow 2\left[\frac{\tan\alpha - \tan 45^0}{1 + \tan\alpha \cdot \tan 45^0}\right] = 1$$

$$\Rightarrow 2\left[\frac{\tan\alpha - 1}{1 + \tan\alpha}\right] = 1$$

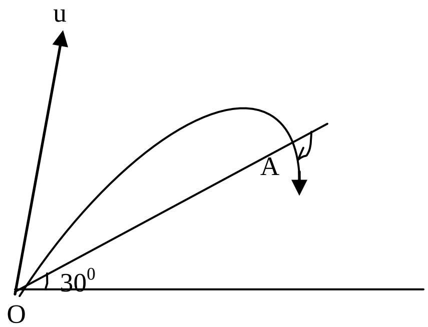
i.e. $2(\tan\alpha - 1) = 1 + \tan\alpha$

$\therefore \tan\alpha = 3$

Problem 17

A particle is projected with speed u so as to strike at right angles a plane through the point of projection inclined at 30^0 to the horizon. Show that the range on this inclined plane is $\frac{4u^2}{7g}$

Solution:



Since u is the velocity of projection, $\beta = 30^0$ is the inclination of the inclined plane, we have proved, Range on the inclined plane = OA

$$= \frac{2u^2 \cdot \sin\beta}{g\left(1+3\sin^2\beta\right)}$$

$$= \frac{2u^2 \cdot \sin 30^0}{g\left(1+3\sin^2 30^0\right)}$$

$$= \frac{2u^2 \times \frac{1}{2}}{g\left(1+\frac{3}{4}\right)} \qquad = \frac{4u^2}{7g}$$

3. 6 Impulsive Forces

3.6.1 Impulse:

The term impulse of force is defined as follows:

(1) The impulse of a constant force F during a time interval T is defined as the product FT.

Let f be the constant acceleration produced on a particle of mass m on which F acts and u, v be respectively the velocity at the beginning and end of the period T.

Then v-u = fT and F = mf.

Hence the impulse I = FT = mfT = m(v-u)

=change of momentum produced.

(2) The impulse of a variable force F during a time interval T is defined to be the time integral of the force for that interval.

i.e. Impulse I = $\int_o^T$ Fdt. This is got as follows. During a short interval of time Δt, the force F can be taken to be constant and hence elementary impulse in this interval = F. Δt. Hence the impulse during the whole time T for which the force F acts is the sum of such impulses and

$$= \underset{\Delta t \to 0}{\mathrm{Lt}} \sum_{t-0}^{T} F.\Delta t = \int_o^T F\,dt.$$

Since F is variable, F = m . $\frac{dv}{dt}$

So impulse = $\int_o^T m \frac{dv}{dt} dt$ = mv – mu, where u and v are the velocities at the beginning and end of the interval and hence this is also equal to the change of momentum produced.

Thus whether a force is a variable or constant,

its impulse = change of momentum produced.

3.6.2 Impulsive Force:

The change of momentum produced by a variable force P acting on a body of mass m from time t = t_1 to t = t_2 is $\int_{t1}^{t2}$ P dt. Suppose P is very large but the time interval t_2 - t_1 during which it acts is very small. It is quite possible that the above definite integral tends to a finite limit. Such a force is called an impulsive force.

Thus an impulsive force is one of large magnitude which acts for a very short period of time and yet produces a finite change of momentum.

Theoretically an impulsive force should be infinitely great and the time during which it acts must be very small. This, of course, is never realized in practice, but approximate examples are (1) the force produced by a hammer-blow (2) the impact of a bullet on a target. In such cases the measurement of the magnitude of the actual force is impracticable but the change in momentum produced may be easily measured. Thus an impulsive force is measured by its impulse i.e. the change of momentum it produces.

Since an impulsive force acts only for a short time on a particle, during this time the distance travelled by a particle having a finite velocity is negligible. Also suppose a body is acted upon by impulsive forces is very short, during this time, the effect of the ordinary finite forces can be neglected.

3.7. Collision of Elastic Bodies

A solid body has a definite shape. When a force is applied at any point of it tending to change its shape, in general, all solids which we meet with in nature yields slightly and get more or less deformed near the point. Immediately, internal forces come into play tending to restore the body to its original form and as soon as the disturbing force is removed, the body regains its original shape. The internal force which acts, when a body tends to recover its original shape after a deformation or compression is called the force of restitution. Also, the properly which causes a solid body to recover its shape is called elasticity. If a body does not tend to recover its shape, it will cause no force of restitution and such a body is said to be inelastic. When a body completely regains its shape after a collision, it is said to be perfectly elastic. If it does not come to its original shape, it is said to be perfectly inelastic.

Definitions:

Two bodies are said to impinge directly when the direction of motion of each before impact is along the common normal at the point where they touch.

Two bodies are said to impinge obliquely, if the direction of motion of either body or both is not along the common normal at the point where they touch.

The common normal at the point of contact is called the line of impact. Thus, in the cause of two spheres, the line of impact is the line joining their centres.

3.8. Fundamental Laws of Impact:

1. Newton's Experimental Law (NEL):

When two bodies impinge directly, their relative velocity after impact bears a constant ratio to their relative velocity before impact and is in the opposite direction. If two bodies impinge obliquely, their relative velocity resolved along their common normal after impact bears a constant ratio to their relative velocity before impact, resolved in the same direction, and is of opposite sign.

The constant ratio depends on the material of which the bodies are made and is independent of their masses. It is generally denoted by e, and is called the *coefficient (or modulus) of elasticity (or restitution or resilience).*

This law can be put symbolically as follows: If u_1, u_2 are the components of the velocities of two impinging bodies along their common normal before impact and v_1, v_2 their component velocities along the same line after impact, all components being measured in the same direction and e is the coefficient of restitution, then

$$\frac{v_2 - v_1}{u_2 - u_1} = -e.$$

The quantity e, which is a positive number, is never greater than unity. It lies between 0 and 1. Its value differs widely for different bodies; for two glass balls, one of lead and the other of iron, its value is about 0.13. Thus, when one or both the bodies are altered, e becomes different but so long as both the bodies remain the same, e is constant. Bodies for which e = 0 are said to be inelastic. For perfectly elastic bodies, e=1. Probably, there are no bodies in nature coming strictly under wither of these headings. Newton's law is purely empirical and is true only approximately, like many experimental laws.

2. Motion of two smooth bodies perpendicular to the line of Impact:

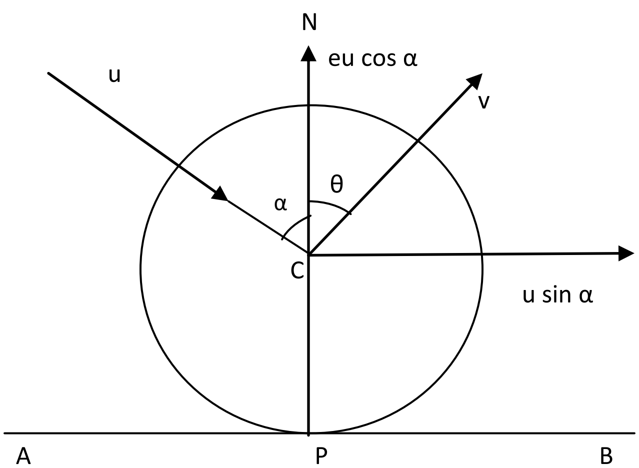
When two smooth bodies impinge, the only force between them at the time of impact is the mutual reaction which acts along the common normal. There is no force acting along the common tangent and hence there is no change of velocity in that direction. Hence the velocity of either body resolved in a direction perpendicular to the line of impact is not altered by impact.

3. Principle of Conservation of Momentum (PCM) :

We can apply the law of conservation of momentum in the case of two impinging bodies. The algebraic sum of the momenta of the impinging bodies after impact is equal to the algebraic sum of their moments before impact, all momenta being measured along the common normal.

3.9. Impact of a smooth sphere on a fixed smooth plane:

A smooth sphere, or particle whose mass is m and whose coefficient of restitution is e, impinges obliquely on a smooth fixed plane; to find its velocity and direction of motion after impact.



Let AB be the plane and P the point at which the sphere strikes it. The common normal at P is the vertical line at P passing through the centre of the sphere. Let it be PC. This is the line of impact. Let the velocity of the sphere before impact be u at an angle α with CP and v its velocity after impact at an angle θ with CN as shown in the figure.

Since the plane and the sphere are smooth, the only force acting during impact is the impulsive reaction and this is along the common normal. There is no force parallel to the plane during impact. Hence the velocity of the sphere, resolved in a direction parallel to the plane is unaltered by the impact.

Hence $v \sin\theta = u \sin\alpha$ … (1)

By Newton's experimental law, the relative velocity of the sphere along the common normal after impact is (-e) time its relative velocity along the common normal before impact. Hence

$$v \cos \theta - 0 = -e(-u \cos \alpha - 0)$$

i.e. $v \cos \theta = eu \cos \alpha$ …(2)

Squaring (1) and (2), and adding, we have

$v^2 = u^2 (\sin^2 \alpha + e^2 \cos^2 \alpha)$

i.e. $v = u\sqrt{\sin^2\alpha + e^2\cos^2\alpha}$ ... (3)

Dividing (2) by (1), we have $\cot \theta = e \cot \alpha$ ... (4)

Hence the (3) and (4) give the velocity and direction of motion after impact.

Corollary 1: If e = 1, we find that from (3) v = u and from (4) $\theta = \alpha$. Hence if a perfectly elastic sphere impinges on a fixed smooth plane, its velocity is not altered by impact and the angle of reflection is equal to the angle of incidence.

Cor. 2: If e = 0, then from (2), $v \cos \theta = 0$ and from (3), $v = u \sin \alpha$. Hence $\cos \theta = 0$ i.e. $\theta = 90^o$. Hence the inelastic sphere slides along the plane with velocity $u \sin \alpha$

Cor. 3: If the impact is direct we have $\alpha = 0$. Then $\theta = 0$ and from (3) v=cu. Hence if an elastic sphere strikes a plane normally with velocity u, it will rebound in the same direction with velocity eu.

Cor. 4: The impulse of the pressure on the plane is equal and opposite to the impulse of the pressure on the sphere. The impulse I on the sphere is measured by the change in momentum of the sphere along the common normal.

$I = mv \cos \theta - (- mu \cos \alpha)$

$= m (v \cos \theta + u \cos \alpha)$

$= m (cu \cos \alpha + u \cos \alpha)$

$= mu \cos \alpha (1 + e)$

Cor. 5: Loss of kinetic energy due to the impact

$$= \frac{1}{2}mu^2 - \frac{1}{2}mv^2 = \frac{1}{2}mu^2 - \frac{1}{2}mu^2\ (\sin^2\alpha + e^2\cos^2\alpha)$$

$$= \frac{1}{2}mu^2\,(1 - \sin^2\alpha + e^2\cos^2\alpha)$$

$$= \frac{1}{2}mu^2\,(\cos^2\alpha - e^2\cos^2\alpha)$$

$$= \frac{1}{2}(1 - e^2)\,mu^2\cos^2\alpha$$

If the sphere is perfectly elastic, e = 1 and the loss of kinetic energy is zero.

Problem 18

A particle falls from a height h upon a fixed horizontal plane: if e be the coefficient of restitution, show that the whole distance described before the particle has finished rebounding is $h\left(\frac{1+e^2}{1-e^2}\right)$. Show also that the whole time taken is $\frac{1+e}{1-e}.\sqrt{\frac{2h}{g}}$.

Solution:

Let u the velocity of the particle on first hitting the plane. Then $u^2 = 2gh$. After the first impact, the particle rebounds with a velocity eu and ascends a certain height, retraces its path and makes a second impact with the plane with velocity eu. After the second impact, it rebounds with a velocity c^2u and the process is repeated a number of times. The velocities after the third, fourth etc. impacts are e^3u e^4u etc.

The height ascended after the first impact with velocity eu is $\frac{(\text{velocity})^2}{2g}$

$$= \frac{e^2u^2}{2g}$$

The height ascended after the second impact with velocity e^2u is $e^4u^2/2g$ and so on.

∴ Total distance travelled before the particle stops rebounding

$$= h + 2\left(\frac{e^2u^2}{2g} + \frac{e^4u^2}{2g} + \frac{e^6u^2}{2g} + \dots\dots\dots\right)$$

$$= h + \frac{2\,.\,e^2u^2}{2g}\ (1 + e^2 + e^4 + \dots\dots\dots \text{ to } \infty)$$

$$= h + \frac{e^2 u^2}{g} \cdot \frac{1}{1-e^2}$$

$$= h + \frac{e^2 .2gh}{g} \cdot \frac{1}{1-e^2}$$

$$= h\,(1 + \frac{2e^2}{1-e^2})$$

$$= h. \frac{(1+e^2)}{(1-e^2)}$$

Considering the motion before the first impact, we have the initial velocity = 0, acceleration = g, final velocity = u and so if t is the time taken, u = 0 + gt.

$$\therefore t = \frac{u}{g} = \frac{\text{velocity}}{g}$$

Time interval between the first and second impacts is

= 2 x time taken for gravity to reduce the velocitiy to 0.

= 2. velocity / g

= 2 eu / g.

Similarly time interval between the second and third impacts

= 2 e^2 u/g and so on.

So total time taken

$$= \frac{u}{g} + 2(\frac{eu}{g} + \frac{e^2 u}{g} + \frac{e^3 u}{g} + \ldots \ldots \infty)$$

$$= \frac{u}{g} + \frac{2\,e\,u}{g}\;(1 + e + e^2 + \ldots \ldots \ldots \text{to } \infty)$$

$$= \frac{u}{g} + \frac{2\,e\,u}{g} \cdot \frac{1}{1-e} = \frac{u}{g}[\,1 + \frac{2e}{1-e}\,]$$

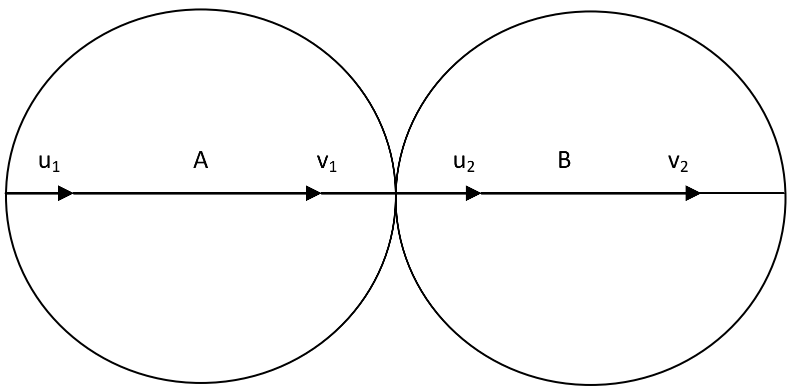
$$= \frac{u}{g} + (\frac{1+e}{1-e})$$

$$= \frac{\sqrt{2gh}}{g}(\frac{1+e}{1-e}) = \left(\frac{1+e}{1-e}\right)\sqrt{\frac{2h}{g}}.$$

3.10 Direct impact of two smooth spheres:

A smooth sphere of mass m_1 impinges directly with velocity u_1 on another smooth sphere of mass m_2, moving in the same direction with velocity u_2. If the coefficient of restitution is e, to find their velocities after the impact:

Solution:



AB is the line of impact, i.e. the common normal. Due to the impact there is no tangential force and hence, for either sphere the velocity along the tangent is not altered by impact. But before impact, the spheres had been moving only along the line AB (as this is a case of direct impact). Hence for either sphere tangential velocity after impact = its tangent velocity before impact = 0. So, after impact, the spheres will move only in the direction AB. Let their velocities be v_1 and v_2.

By Newton's experimental law, the relative velocity of m_2 with respect to m_1 after impact is (-e) times the corresponding relative velocity before impact.

$\therefore v_2 - v_1 = -e\,(u_2 - u_1)$ …….(1)

By the principle of conservation of momentum, the total momentum along the common normal after impact is equal to the total momentum in the same direction before impact.

$\therefore m_1 v_1 + m_2 v_2 = m_1 u_1 + m_2 u_2$ …….(2)

(2) – (1) x m_2 gives

$v_1 (m_1 + m_2) = m_1 u_1 + m_2 u_2 + em_2 (u_2 - u_1)$

$= m_2 u_2 (1 + e) + (m_1 - em_2) u_1$

$\therefore v_1 = \frac{m_2 u_2 (1+e) + (m_1 - em_2) u_1}{m_1 + m_2}$ ... (3)

(1) x m_1 + (2) gives

$$v_2 (m_1 + m_2) = - em_1 (u_2 – u_1) + m_1u_1 + m_2u_2$$

$$= m_1u_1 (1 + e) + (m_2 – em_1) u_2$$

$$\therefore v_2 = \frac{m_1 u_1 (1 + e) + (m_2 - em_1) u_2}{m_1 + m_2} \quad \text{... (4)}$$

Equations (3) and (4) give the velocities of the spheres after impact.

Note: If one sphere say m_2 is moving originally in a direction opposite to that of m_1, the sign of u_2 will be negative. Also it is most important that the directions of v_1 and v_2 must be specified clearly. Usually we take the positive direction as from left to right and then assume that both v_1 and v_2 are in this direction. If either of them is actually in the opposite direction, the value obtained for it will turn to be negative.

In writing equation (1) corresponding to Newton's law, the velocities must be subtracted in the same order on both sides. In all problems it is better to draw a diagram showing clearly the positive direction and the directions of the velocities of the bodies.

Corollary 1. If the two spheres are perfectly elastic and of equal mass, then e = 1 and $m_1 = m_2$. Then, from equations (3) and (4), we have

$$v_1 = \frac{m_1 u_2 . 2 + 0}{2m_1} = u_2 \text{ and } v_2 = \frac{m_1 u_1 . 2 + 0}{2m_1} = u_1.$$

i.e. *If two equal perfectly elastic spheres impinge directly, they interchange their velocities.*

Cor: 2. The impulse of the blow on the sphere A of mass m_1 = change of momentum of A = $m_1 (v_1 – u_1)$.

$$= m_1 \left[\frac{m_2 u_2 (1+e) + m_1 - em_2) u_1}{m_1 + m_2} - u_1\right]$$

$$= m_1 \left[\frac{m_2 u_2 (1+e) + m_1 u_1 - em_2 u_1 - m_1 u_1 - m_2 u_1}{m_1 + m_2}\right]$$

$$= \frac{m_1 [m_2 u_2 (1+e) - m_2 u_1 (1+e)]}{m_1 + m_2}$$

$$= \frac{m_1 \ m_2 \,(1+e)\,(u_2 - u_1)}{m_1 + m_2}$$

The impulsive blow on m_2 will be equal and opposite to the impulsive blow on m_1.

Loss of kinetic energy due to direct impact of two smooth spheres:

Two spheres of given masses with given velocities impinge directly; to show that there is a loss of kinetic energy and to find the amount:

Let m_1 m_2 be the masses of the spheres, u_1 and u_2, v_1 and v_2 be their velocities before and after impact and e the coefficient of restitution.

By Newton's law, $v_2 - v_1 = -e\,(u_2 - u_1)$ … (1)

By the principle of conservation of momentum,

$m_1v_1 + m_2v_2 = m_1u_1 + m_2u_2$ ….(2)

Total kinetic energy before impact

$$= \frac{1}{2} m_1 u_1^{\ 2} + \frac{1}{2} m_2u_2^2$$

and total kinetic energy after impact

$$= \frac{1}{2} m_1v_1^2 + \frac{1}{2} m_2v_2^2$$

Change in K.E. = initial K.E. – final K.E.

$$= \frac{1}{2} m_1u_1^2 + \frac{1}{2} m_2u_2^2 - \frac{1}{2} m_1v_1^2 - \frac{1}{2} m_2v_2^2$$

$$= \frac{1}{2} m_1 (u_1 - v_1)(u_1 + v_1) + \frac{1}{2} m_2 (u_2 - v_2)(u_2 + v_2)$$

$$= \frac{1}{2} m_1 (u_1 - v_1)(u_1 + v_1) + \frac{1}{2} m_1 (v_1 - u_1)(u_2 + v_2)$$

$[\because \ \ m_2 (u_2 - v_2) \ = \ m_1 (v_1 - u_1)$ from (2)]

$$= \frac{1}{2} m_1 (u_1 - v_1)\,[u_1 - u_2 \ - \ (v_2 - v_1)\,]$$

$$= \frac{1}{2} m_1 (u_1 - v_1)\,[u_1 - u_2 + e\ (u_2 - u_1)] \ \text{ using (1)}$$

$$= \frac{1}{2} m_1 (u_1 - v_1)(u_1 - u_2)\,(1 - e) \qquad \text{…..(3)}$$

Now, from (2), $m_1 (u_1 - v_1) = m_2 (v_2 + u_2)$

$\therefore \ \frac{u_1 - v_1}{m_2} = \frac{v_2 - u_2}{m_1}$ and each $= \frac{u_1 - v_1 + v_2 - u_2}{m_1 + m_2}$

i.e. each $= \frac{(u_1 - u_2) + (v_2 - v_1)}{m_1 + m_2}$

$$= \frac{(u_1 - u_2) - e\,(u_2 - u_1)}{m_1 + m_2} \quad \text{using (1)}$$

$$= \frac{(u_1 - u_2)\ (1 + e)}{m_1 + m_2}$$

$\therefore u_1 - v_1 = \dfrac{m_2\,(u_1 - u_2)\ (1 + e)}{m_1 + m_2}$ and substituting this in (3),

$$\text{Change in K.E.} = \frac{1}{2}\ \frac{m_1\, m_2\,(u_1 - u_2)\,(1 + e)\,(u_1 - u_2)\ (1 - e)}{m_1 + m_2}$$

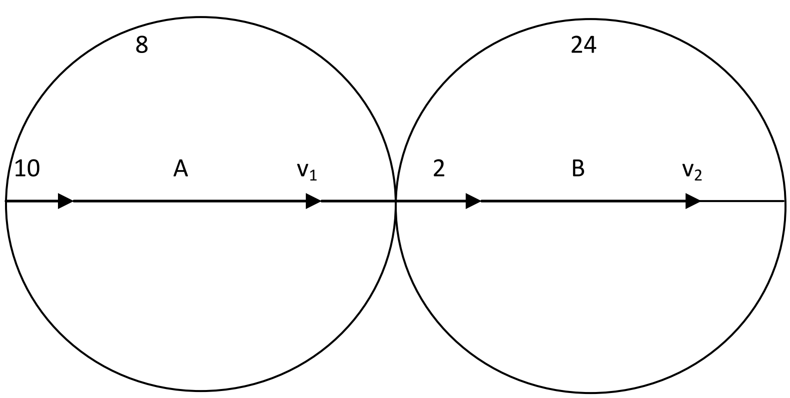
$$= \frac{1}{2}\ \frac{m_1\, m_2\,(u_1 - u_2)^2\,(1 - e^2)}{m_1 + m_2} \qquad \ldots(4)$$

As e < 1, the expression (4) is always positive and so the initial K.E. of the system is greater than the final K.E. So there is actually a loss of total K.E. by a collision. Only in the case, when e=1, i.e. only when the bodies are perfectly elastic, the expression (4) becomes zero and hence the total K.E. is unchanged by impact.

Problem 19

A ball of mass 8 gm. moving with a velocity of 10 cm. per sec. impinges directly on another of mass 24 gm., moving at 2cm per sec. in the same direction. If e = ½, find the velocities after impact. Also calculate the loss in kinetic energy.

Solution:



Let v_1 and v_2 cm. per sec. be the velocities of the masses 8gm and 24 gm respectively after impact.

By Newton's Law, v_2-v_1 = - $\frac{1}{2}$ (2-10) = 4 …… (1)

By the principle of momentum,

$24v_2 + 8v_1 = 24 \times 2 + 8 \times 10 = 128$

i.e. $3v_2 + v_1 = 16$

Solving (1) and (2), v_1 = 1 cm. / sec., v_2 = 5 cm./ sec.

The K.E. before impact = $\frac{1}{2} . 8.10^2 + \frac{1}{2} . 24.2^2$

= 448 dunes

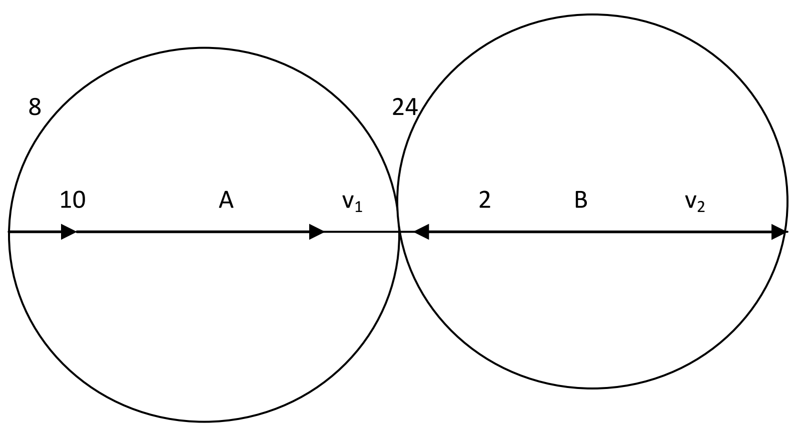
The K.E. after impact = $\frac{1}{2} . 8.1^2 + \frac{1}{2} . 24.5^2$ = 304 dines

∴ Loss in K.E. = 144 dynes

Problem 20

If the 24 gm.mass in the previous question be moving in a direction opposite to that of the 8 gm. mass, find the velocities after impact.

Solution:



Let v_1 and v_2 cm/sec. be the velocities of the 8gms and 24 gms mass respectively after impact.

By Newton's law,

$V_2 - v_1 = -\frac{1}{2} (-2 - 10) = 6$ ……… (1)

By conservation of momentum,

$24v_2 + 8v_1 = 24 \text{ x } (-2) + 8 \text{ x } 10 = 32$ i.e. $3v_2 + v_1 = 4$ ………….. (2)

Solving (1) and (2), $v_1 = -\frac{7}{2}$ cm/sec $v_2 = \frac{1}{2}$ cm / sec.

The negative sign of v_1 shows that the direction of motion of the 8 gm. Mass is reversed, as we had taken the direction left to right as positive and assumed v1 to be in this direction. Since v2 is positive, the 24gm. ball moves from left to right after impact, so that its direction of motion is also reversed.

Problem 21

A ball overtakes another ball of m times its mass, which is moving with $\frac{1}{n}$ *th of its velocity in the same direction. If the impact reduces the first ball to rest, prove that the coefficient of elasticity is* $\frac{m+n}{m\ (n-1)}$

Deduce that $m > \frac{n}{n-2}$

Taking AB as positive direction (as shown in the previous diagram), let the mass of the first ball be k and u its velocity along AB before impact. Then, for the second ball, the mass is mk and $\frac{u}{n}$ is the velocity before impact. After impact, the first ball is reduced to rest and let v be the velocity of the second ball.

By Newton's law of impact, we have

$v - 0 = -e.\ (\frac{u}{n} - u)$ i.e. $v = \frac{eu(n-1)}{n}$ .. (1)

By principle of conservation of momentum along AB,

$K \text{ x } 0 + mk.\ V = ku + mk.\ \frac{1}{n} u$

i.e. $mv = u + \frac{m}{u} u = \frac{u\ (m+n)}{n}$ ..(2)

Substituting value of v from (1) in (2), 12 have

$\frac{meu\ (n-1)}{n} = \frac{u(m+n)}{n}$ or $e = \frac{(m+n)}{m\ (n-1)}$

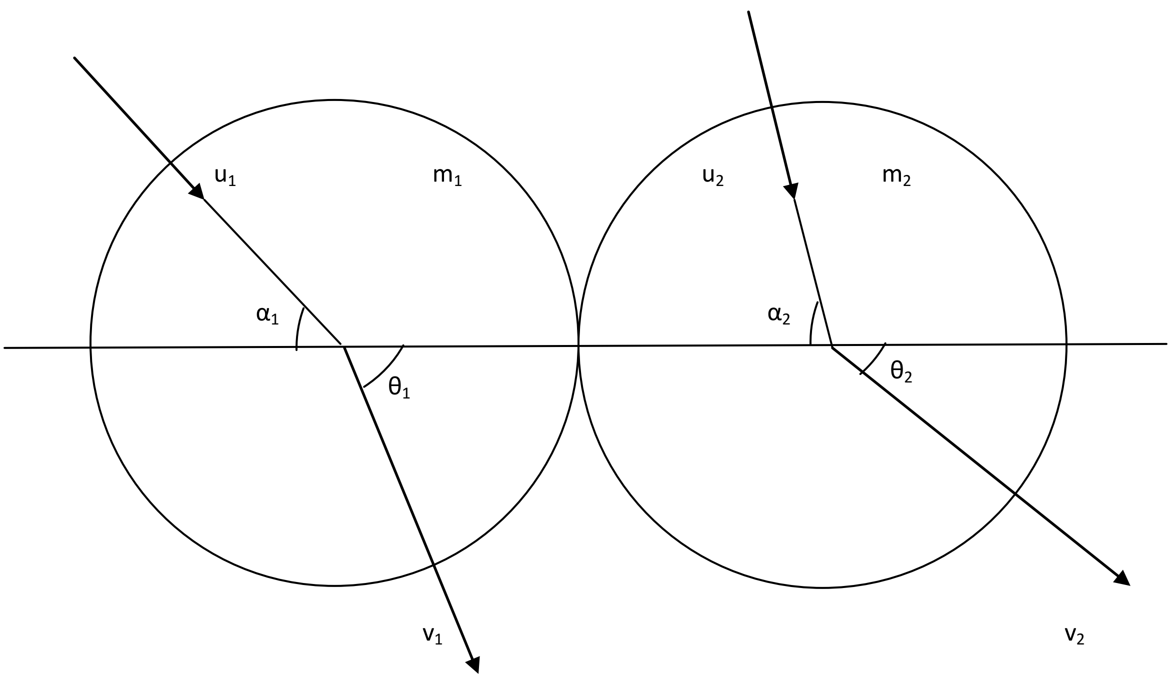
Now e is positive and less than 1.

$\therefore$ m (n – 1) > m + n i.e. mn – 2m > n

$\therefore$ m (n – 2) > n or m > $\frac{n}{n-2}$

3.11 Oblique impact of two smooth spheres:

A smooth sphere of mass m_1 impinges obliquely with velocity u_1 on another smooth sphere of mass m_2 moving with velocity u_2. If the directions of motion before impact make angles α_1 and α_2 respectively with line joining the centres of the spheres and if the coefficient of restitution be e, to find the velocities and directions of motion after impact.



Let the velocities of the spheres after impact be v_1 and v_2 in directions inclined at angles θ_1 and θ_2 respectively to the line of centres. Since the spheres are smooth, there is no force perpendicular to the line of centres and therefore, for each sphere the velocities in the tangential direction are not affected by impact.

$\therefore$ v1 sin θ1 = u1 sin α1 … (1) and

$v_2 \sin \theta_2 = u_2 \sin \alpha_2$ … (2)

By Newton's law concerning velocities along the common normal AB,

$v_2 \cos \theta_2 - v_1 \cos \theta_1 = -e (u_2 \cos \alpha_2 - u_1 \cos \alpha_1)$ ... (3)

By the principle of conservation of momentum along AB,

$m_{2.}\, v_2 \cos\theta_2 + m_1 v_1 \cos\theta_1 = m_2 u_2 \cos\alpha_2 + m_1 u_1 \cos\alpha_1$ ….(4)

(4) – (3) x m2 gives

$v_1 \cos\theta_1. (m_1 + m_2) = m_2 u_2 \cos\alpha_2 + m_1 u_1 \cos\alpha_1$

$+ em_2 (u_2 \cos\alpha_2 - u_1 \cos\alpha_1)$

i.e. $v_1 \cos\theta_1 = \dfrac{u_1 \cos\ \alpha_1 (m_1 - em_2) + m_2 u_2 \cos\ \alpha_2\ (1+e)}{m_1 + m_2}$ … (5)

(4) + (3) x m_1 gives

$V_2 \cos\theta_2 = \dfrac{u_2 \cos\ \alpha_2 (m_2 - em_1) + m_1 u_1 \cos\ \alpha_1\ (1+e)}{m_1 + m_2}$ … (6)

From (1) and (5), by squaring and adding, we obtain v_1^2 and by division, we have $\tan\theta_1$. Similarly from (2) and (6) we get v_2^2 and $\tan\theta_2$. Hence the motion after impact is completely determined.

Corollary 1. If the two spheres are perfectly elastic and of equal mass, then e = 1 and $m_1 = m_2$.

Then from equations (5) and (6) we have

$V_1 \cos\theta_1 = \dfrac{0 + m_1\ u_2\ \cos\ \alpha_2 .2}{2\ m_1} = u_2 \cos\ \alpha_2$

And $V_2 \cos\theta_2 = \dfrac{0 + m_1\ u_1\ \cos\ \alpha_2 .2}{2\ m_1} = u_1 \cos\ \alpha_1$

Hence if two equal perfectly elastic spheres impinge, they interchange their velocities in the direction of the line of centres.

Corollary 2. Usually, in most problems on oblique impact, one of the spheres is at rest. Suppose m_2 is at rest i.e. $u_2 = 0$.

From equation (2), $v_2 \sin\theta_2 = 0$ i.e. $\theta_2 = 0$. Hence m2 moves along AB after impact. This is seen independently, since the only force on m_2 impact is along the line of centres.

Corollary 3:

The impulse of the blow on the sphere A of mass m_1

= change of momentum of A along the common normal

$= m_1 \ (v_1 \ \cos\theta_1 \ - \ u_1 \ \cos\alpha_1)$

$$= m_1 \left[\frac{u_1 \cos \alpha_1 (m_1 - em_2) + m_2 u_2 \cos \alpha_2 \ (1+e)}{m_1 + m_2} - u_1 \cos \alpha_1\right]$$

$$= m_1 \left[\frac{m_1 u_1 \cos \alpha_1 - em_2 u_1 \cos \alpha_1 + m_2 u_2 \cos \alpha_2 + em_2 u_2 \cos \alpha_2 - m_1 u_1 \cos \alpha_1 - m_2 u_1 \cos \alpha_1}{m_1 + m_2}\right]$$

$$= \frac{m_1 \ [m_2 u_2 \cos \alpha_2 (1+e) - m_2 u_1 \cos \alpha_1 (1+e)]}{m_1 + m_2}$$

$$= \frac{m_1 \ m_2 (1+e)}{m_1 + m_2} (u_2 \cos \alpha_2 - u_1 \cos \alpha_1)$$

The impulsive blow on m_2 will be equal and opposite to the impulsive blow on m_1.

Loss of kinetic energy due to oblique impact of two smooth spheres:

Two spheres of masses m_1 and m_2 moving with velocities u_1 and u_2 at angles α_1 and α_2 with their line of centres, come into collision. To find an expression for the loss of kinetic energy:

The velocities perpendicular to the line of centres are not altered by impact. Hence the loss of kinetic energy in the case of oblique impact is therefore the same as in the case of direct impact if we replace in the expression (4) on page 236, the quantities u_1 and u_2 by $u_1 \cos \alpha_1$ and $u_2 \cos \alpha_2$ respectively.

Therefore the loss is $= \frac{1}{2} \frac{m_1 \ m_2}{m_1 + m_2} (1 - e^2) \ (u_1 \cos \alpha_1 - u_2 \cos \alpha_2)^2$

We shall now derive this independently.

Let v_1 and v_2 be the velocities of the spheres after impact, in directions inclined at angles θ1 and θ2 respectively to the line of centres. As explained in § 8.7 the tangential velocity of each sphere is not altered by impact.

$\therefore v_1 \sin \theta_1 = u_1 \sin \alpha_1$ … (1) and $v_2 \sin \theta_2 = u_2 \sin \alpha_2$ … (2)

By Newton's of rule

$v_2 \cos\theta_2 - v_1 \cos\theta_1 = - e\,(u_2 \cos\alpha_2 - u_1 \cos\alpha_1)$ … (3)

By conservation of momenta,

$m_2 v_2 \cos\theta_2 + m_1 v_1 \cos\theta_1 = m_2 u_2 \cos\alpha_2 + m_1 u_1 \cos\alpha_1$

i.e. $m_1 (u_1 \cos\alpha_1 - v_1 \cos\theta_1) = m_2 (v_2 \cos\theta_2 - u_2 \cos\alpha_2)$ … (4)

Change in K.E.

$$= \frac{1}{2} m_1 u_1^2 + \frac{1}{2} m_2 u_1^2 - \frac{1}{2} m_1 v_1^{\;2} - \frac{1}{2} m_2 v_2^{\;2}$$

$$= \frac{1}{2} m_1 u_1^2 (\cos^2\alpha_1 + \sin^2\alpha_1) + \frac{1}{2} m_2 u_2^{\;2} (\cos^2\alpha_2 + \sin^2\alpha_2)$$

$$- \frac{1}{2} m_1 v_1^2 (\cos^2\theta_1 + \sin^2\theta_1) - \frac{1}{2} m_2 v_2^2 (\cos^2\theta_2 + \sin^2\theta_2)$$

$$= \frac{1}{2} m_1 u_1^2 \cos^2\alpha_1 + \frac{1}{2} m_2 u_2^2 \cos^2\alpha_2 - \frac{1}{2} m_1 v_1^2 \cos^2\theta_1$$

$$- \frac{1}{2} m_2 v_2^2 \cos^2\theta_2 \text{ using (1) and (2)}$$

$$= \frac{1}{2} m_1 (u_1^2 \cos\alpha_1 - v_1^2 \cos^2\theta_1) + \frac{1}{2} m_2 (u_2^2 \cos^2\alpha_2 - v_2^2 \cos^2\theta_2)$$

$$= \frac{1}{2} m_1 (u_1 \cos\alpha_1 + v_1 \cos\theta_1)(u_1 \cos\alpha_1 - v_1 \cos\theta_1)$$

$$+ \frac{1}{2} m_2 (u_2 \cos\alpha_2 + v_2 \cos\theta_2)(u_2 \cos\alpha_2 - v_2 \cos\theta_2)$$

$$= \frac{1}{2} m_1 (u_1 \cos\alpha_1 + v_1 \cos\theta_1)(u_1 \cos\alpha_1 - v_1 \cos\theta_1)$$

$$- \frac{1}{2} (u_2 \cos\alpha_2 + v_2 \cos\theta_2).\ m_1 (u_1 \cos\alpha_1 - v_1 \cos\theta_1) \text{ using (4)}$$

$$= \frac{1}{2} m_1 (u_1 \cos\alpha_1 - v_1 \cos\theta_1)(u_1 \cos\alpha_1 + v_1 \cos\theta_1 - u_2 \cos\alpha_2 - v_2 \cos\theta_2)$$

$$= \frac{1}{2} m_1 (u_1 \cos\alpha_1 - v_1 \cos\theta_1)\,[u_1 \cos\alpha_1 + u_2 \cos\alpha_2$$

$$+ e\,(u_2 \cos\alpha_2 - u_1 \cos\alpha_1)] \text{ Using (3)}$$

$$= \frac{1}{2} m_1 (u_1 \cos\alpha_1 - v_1 \cos\theta_1)(u_1 \cos\alpha_1 - u_2 \cos\alpha_2)(1 - e) \quad \text{…..(5)}$$

Now from (4),

$$\frac{u_1 \cos\alpha_1 - v_1 \cos\theta_1}{m_2} = \frac{v_2 \cos\theta_2 - u_2 \cos\alpha_2}{m_1}$$

and each $= \dfrac{u_1 \cos\alpha_1 - v_1 \cos\theta_1 + v_2 \cos\theta_2 - u_2 \cos\alpha_2}{m_1 + m_2}$

$$= \frac{(u_1 \cos \alpha_1 - u_2 \cos \alpha_2) + (v_2 \cos \theta_2 - v_1 \cos \theta_1)}{m_1 + m_2}$$

$$= \frac{u_1 \cos \alpha_1 - u_2 \cos \alpha_2 - e \ (u_2 \cos \alpha_2 - u_1 \cos \alpha_1)}{m_1 + m_2} \text{ using (3)}$$

$$= \frac{(u_1 \cos \alpha_1 - u_2 \cos \alpha_2)(1+e)}{m_1 + m_2}$$

$$\therefore u_1 \cos \alpha_1 - v_1 \cos \theta_1 = \frac{m_2(1+e)}{m_1+m_2} (u_1 \cos \alpha_1 - u_2 \cos \alpha_2)$$

Substituting in (5),

Change in K.E. $= \frac{1}{2} \frac{m_1 m_2 (1+e)}{m_1 + m_2} (u_1 \cos \alpha_1 - u_2 \cos \alpha_2)$

$$\times (u_1 \cos \alpha_1 - u_2 \cos \alpha_2)(1+e)$$

$$= \frac{1}{2} \frac{m_1 m_2}{m_1 + m_2} (1 - e^2)(u_1 \cos \alpha_1 - u_2 \cos \alpha_2)^2$$

If the spheres are perfectly elastic, e = 1 and the loss of kinetic energy is zero.

Problem 22

A ball of mass 8 gms. moving with velocity 4 cms. Per sec. impinges on a ball os mass 4 gms. Moving with velocity 2 cm. per sec. If their velocities before impact be inclined at angle 30^o and 60^o to the joining their centres at the moment of impact, find their velocities after impact when e = $\frac{1}{2}$

Solution:

In the diagram in the oblique impact of two smooth spheres, let m_1 = 8 u_1 =4 $\alpha_1 = 30^0$, m_2 =4, u_2= 2, $\alpha_2 = 60^0$

Let the velocities of the spheres after impact be v_1 and v_2 in directions inclined at angles θ_1 and θ_2 respectively to the line of centres.

The tangential velocity of each sphere is not affected by impact

$\therefore v_1 \sin \theta_1 = 4 \sin 30^o = 2$ …(1)

and $v_2 \sin \theta_2 = 2 \sin 60^o = \sqrt{3}$ …(2)

By Newton's Law,

$v_2 \cos \theta_2 - v_1 \cos \theta_1 = - e (2 \cos 60^o - 4 \cos 30^o)$

$= - \frac{1}{2} (2. \frac{1}{2} - 4. \frac{\sqrt{3}}{2})$

$\frac{1}{2} (2\sqrt{3} - 1)$ …(3)

By conservation of momenta along AB,

$4v_2 \cos \theta_2 + 8v_1 \cos \theta_1 = 4 \text{ x } 2 \cos 60^o + 8 \text{ x } 4 \cos 30^o = 4 + 16\sqrt{3}$

i.e. $v_2 \cos \theta_2 + 2v_1 \cos \theta_1 = 1 + 4\sqrt{3}$ …(4)

$\therefore 3v_1 \cos \theta_1 = 1 + 4\sqrt{3} - \frac{1}{2}(2\sqrt{3} - 1) = \frac{3 + 6\sqrt{3}}{2}$

i.e. $v_1 \cos \theta^1 = \frac{1+ 2\sqrt{3}}{2}$ …(5)

From (4), $v_2 \cos \theta_2 = 1 + 4\sqrt{3} - 1 - 2\sqrt{3} = 2\sqrt{3}$ …(6)

From (1) and (5), $v_1^2 = 2^2 + (\frac{1+ 2\sqrt{3}}{2})^2$

$= 4 + \frac{1+ 4\sqrt{3} + 12}{4} = \frac{29 + 4\sqrt{3}}{4}$

$\therefore v_1 = \frac{29- 4\sqrt{3}}{2}$ cm. per sec.

Dividing (1) by (5), $\tan \theta 1 = \frac{4}{1+2\sqrt{3}}$

From (2) and (6)

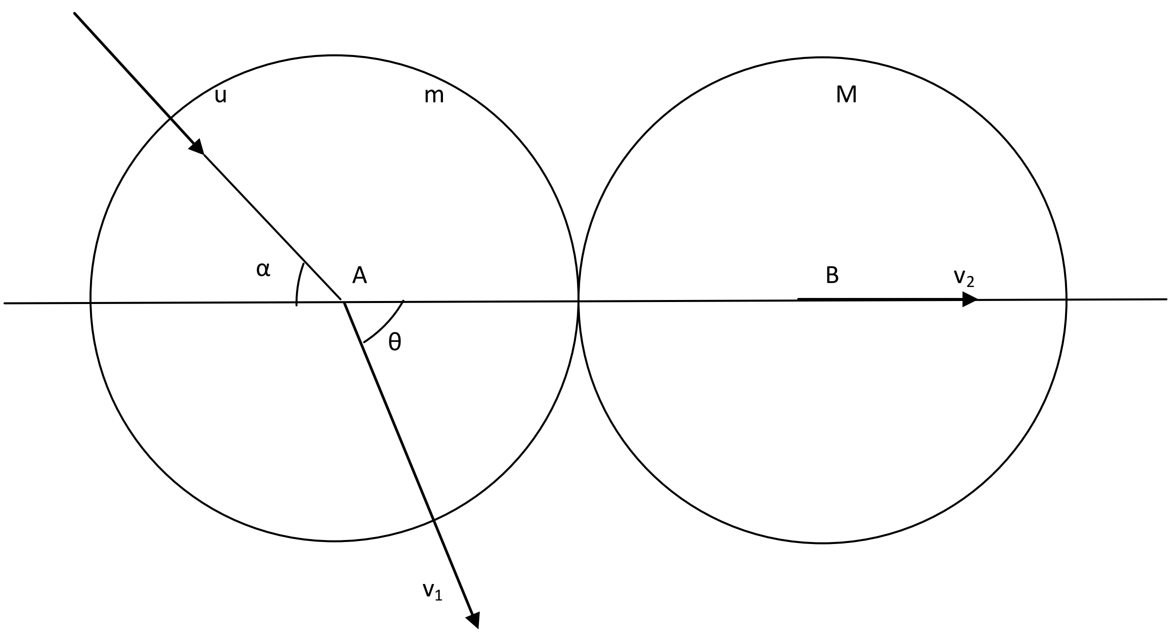
$v_2^2 = 3 + 12 = 15$ and $\therefore v_2 = \sqrt{15}$ cm / sec

Dividing (2) by (6), $\tan \theta_2 = \frac{1}{2}$

Problem 23

A smooth sphere of mass m impinges obliquely on a smooth sphere of mass M which is at rest. Show that if m = eM, the directions of motion after impact are at right angles. (e is the coefficient of restitution)

Solution:



Considering the sphere M, its tangential velocity before impact is zero and hence after impact also, its tangential velocity is zero.

($\because$ During impact, there is no force acting along the common tangent).

Hence, after impact, M will move along AB. Let its velocity be v_2. Let the velocity of m be v_1 at an angle θ to AB, after impact.

By Newton's rule $v_2 - v_1 \cos\theta = -e(0 - u\cos\alpha)$

i.e. $v_2 - v_1 \cos\theta = eu\cos\alpha$ ...(1)

By conservation of momenta along AB,

$M.\, v_2 + m\, v_1 \cos\theta = M.\,0 + m.\, u\cos\alpha$ …(2)

Multiplying (1) by M and subtracting from (2),

$mv_1 \cos\theta + M v_1 \cos\theta = mu \cos\alpha - M eu \cos\alpha$

i.e. $v_1 \cos\theta = \frac{u \cos \alpha \,(m - eM)}{m+M} = \frac{u \cos \alpha.0}{m+M}$ ($\because$ m = e M)

$= 0$

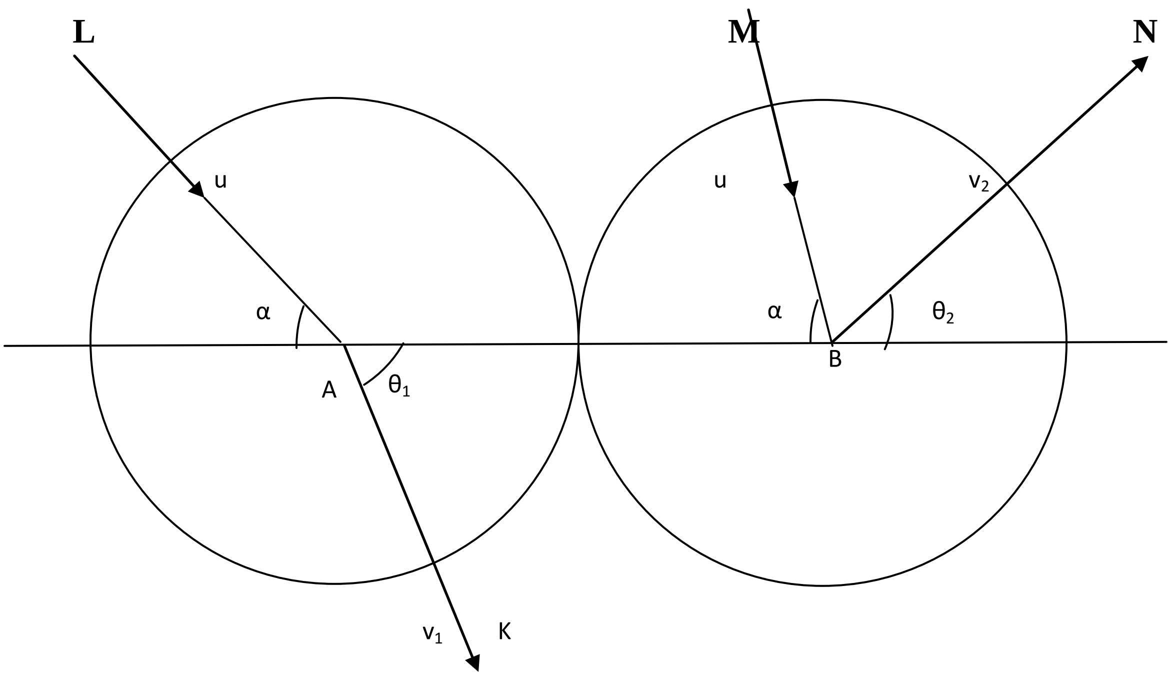
$\therefore \cos\theta = 0$ or $\theta = 90^o$

i.e. The direction of motion of m is perpendicular to AB.

Problem 24

Two equal elastic balls moving in opposite parallel direction with equal speeds impinge on one another. If the inclination of their direction of motion to the line of centres be $\tan^{-1}(\sqrt{e})$ *where e is the coefficient of restitution, show that their direction of motion will be turned through a right angle.*

Solution:



Let m be the mass of either sphere: AB is the line of impact. Before impact, the directions of motion are LA and BM making the same acute angle α with AB as shown in the figure. Let u be their velocity.

After impact, let the sphere A proceed in the direction AK with velocity v_1 at an angle θ_1 to AB and the sphere B proceed in the direction BN with velocity v_2 at an angle θ_2 to AB.

The tangential velocity of either sphere is not affected by impact.

$\therefore v_1 \sin\theta_1 = u \sin\alpha$ … (1) and

$v_2 \sin\theta_2 = u \sin\alpha$ …(2)

By Newton's Law, (resolving all velocities along AB),

$v_2 \cos\theta_2 - v_1 \cos\theta_1 = -e(-u\cos\alpha - u\cos\alpha)$

i.e. $v_2 \cos\theta_2 + v_1 \cos\theta_1 = 2eu\cos\alpha$ … (3)

By conservation of momenta along AB,

$m(v_2\cos\theta_2) + m.v_1\cos\theta_1 = m(-u\cos\alpha) + mu\cos\alpha$

i.e. $v_2\cos\theta_2 + v_1\cos\theta_1 = 0$ …(4)

(4) – (3) gives $v_1\cos\theta_1 = -2eu\cos\alpha$

$\therefore v_1\cos\theta_1 = -eu\cos\alpha$ …(5)

From (4), $v_2\cos\theta_2 = -v_1\cos\theta_1 = eu\cos\alpha$ …(6)

Dividing (1) by (5),

$\tan\theta_1 = -\frac{1}{e}\tan\alpha = -\frac{1}{e}\sqrt{e}$ ($\because \alpha = \tan^{-1}\sqrt{e}$ given)

$= -\frac{1}{\sqrt{e}} = -\frac{1}{\tan\alpha} = -\cot\alpha = \tan(90^o + \alpha)$

$\therefore \theta_1 = -90^o + \alpha$

Dividing (2) by (6), $\tan\theta_2 = \frac{1}{e}\tan\alpha = \cot\alpha = \tan(90^o - \alpha)$

$\therefore \theta_2 = 90^o - \alpha$.

Hence their directions of motion are turned through a right angle.

UNIT IV

SIMPLE HARMONIC MOTION

Simple Harmonic Motion (S.H.M) is an interesting special type of motion in nature, having forward and backward oscillation (or) to and fro oscillation about a fixed point. The fixed point is known as the mean position or equilibrium position. When the oscillation is very small we prove the motion is simple harmonic. In this section we study about the resultant of two S.H.M'S of the same period in the same straight line and in two perpendicular lines. Also we find the periodic time of oscillation of a simple pendulum.

Examples

Small oscillation of a cradle, simple pendulum, seconds pendulum, simple equivalent pendulum, transverse vibrations of a plucked violin string etc.

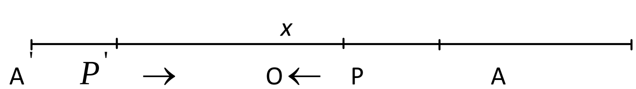
Hooke's law

Tension of an elastic string or spring is directly proportional to its extended length and indirectly proportional to its natural length.

4.1 Simple Harmonic Motion in a straight line

Definition

When a particle moves in a straight line so that its acceleration is always directed towards a fixed point in the line and proportional to the distance from that point, its motion is called Simple Harmonic Motion.



Let O be a fixed point on the straight line A^1 OA on which a particle is having simple harmonic motion. Take O as the origin and OA as the X axis. Let P be the position of the particle at time t such that OP = x. The magnitude of the acceleration at P is μx where μ is a positive constant. The acceleration at P in the positive direction of the X axis is $-\mu$ x towards O.

Hence the equation of motion of P is $\frac{d^2x}{dt^2} = -\mu x$ ……..(1)

Equation (1) is the fundamental differential equation representing a S.H.M.

If v is the velocity of the particle at time t (1) can be written as

$v\frac{dv}{dx} = -\mu x$ i.e. $\quad vdv = -\mu x\, dx$ …………(2)

Integrating (2), we have $\frac{v^2}{2} = -\frac{\mu x^2}{2} + c$ ………..(3)

Initially let the particle starts from rest at the point A where OA = a

Hence when x=a, v = 0 = $\frac{dx}{dt}$

Putting these in (3), 0 = $-\frac{\mu a^2}{2}$ +c or c = $\frac{\mu a^2}{2}$

$\therefore v^2 = -\mu x^2 + \mu a^2 = \mu\left(a^2 - x^2\right)$

$\therefore$ v = $\pm\sqrt{\mu\left(a^2 - x^2\right)}$ ………. (4)

Equation (4) gives the velocity v corresponding to any displacement x.

Now as t increases, x decreases. So $\frac{dx}{dt}$ is negative.

Hence we take the negative sign in (4),

$\frac{dx}{dt}$ **= v =** $-\sqrt{\mu\left(a^2 - x^2\right)}$ **……..(5)**

$-\frac{dx}{\sqrt{\left(a^2 - x^2\right)}} = \sqrt{\mu}\, dt$

Integrating, $\cos^{-1}\frac{x}{a} = \sqrt{\mu}\, t + A$

Initially when t = 0, x = a, $\cos^{-1} 1 = 0 + A \Rightarrow$ $\boxed{A=0}$

$\therefore \cos^{-1}\frac{x}{a} = \sqrt{\mu}\, t$ or $\quad x = a\cos\sqrt{\mu}\, t$ ………. (6)

To get the time from A to A^1, put x = $-a$ in (6)

We have $\cos\sqrt{\mu}\, t = -1 = \cos\pi$, t = $\frac{\pi}{\sqrt{\mu}}$

$\therefore$ The time from A to A' and back = $\dfrac{2\pi}{\sqrt{\mu}}$.

Equation (6) can be written as

x = a cos $\sqrt{\mu}$ t = a cos ($\sqrt{\mu}$ t +2π) = a cos ($\sqrt{\mu}$ t + 4 π) etc

= a cos $\sqrt{\mu}$ $\left(t+\dfrac{2\pi}{\sqrt{\mu}}\right)$ = a cos $\sqrt{\mu}\left(t+\dfrac{4\pi}{\sqrt{\mu}}\right)$ etc.

Differentiating (6),

$\dfrac{dx}{dt}=-a\sqrt{\mu}$. sin $\sqrt{\mu}$ t

= $-a\sqrt{\mu}$ sin ($\sqrt{\mu}$ t + 2 π) = $-a$ $\sqrt{\mu}$ sin ($\sqrt{\mu}$ t + 4 π) etc.

= $-a\sqrt{\mu}$ sin $\sqrt{\mu}$ (t + $\dfrac{2\pi}{\sqrt{\mu}}$) = $-a\sqrt{\mu}$ sin $\sqrt{\mu}$ (t + $\dfrac{4\pi}{\sqrt{\mu}}$) etc.

The values of $\dfrac{dx}{dt}$ are the same if t is increased by $\dfrac{2\pi}{\sqrt{\mu}}$ or by any multiple of $\dfrac{2\pi}{\sqrt{\mu}}$. Hence after a time $\dfrac{2\pi}{\sqrt{\mu}}$ the particle is again at the same point moving with the same velocity in the same direction. Hence the particle has the period $\dfrac{2\pi}{\sqrt{\mu}}$.

T = $\dfrac{2\pi}{\sqrt{\mu}}$; frequency = $\dfrac{1}{T}$ = $\dfrac{2\pi}{\sqrt{\mu}}$

The distance through which the particle moves away from the centre of motion on either side of it is called the *amplitude* of the oscillation.

Amplitude = OA = OA' = a.

The periodic time = $\dfrac{2\pi}{\sqrt{\mu}}$, is independent of the amplitude. It depends only on the constant μ which is the acceleration at unit distance from the centre.

Deductions : 1) Maximum acceleration $=\mu.a$ = μ . (amplitude)

2) Since v = $\sqrt{\mu\left(a^2-x^2\right)}$, the greatest value of v is at x = 0 and its

Maximum velocity = a $\sqrt{\mu}$ = $\sqrt{\mu}$. (amplitude) at the centre

General solution of the S.H.M. equation

The S.H.M. equation is $\frac{d^2x}{dt^2} = -\mu x$

i.e. $\frac{d^2x}{dt^2} + \mu x = 0$ …….(1)

(1) is a differential equation of the second order with constant coefficients. Its general solution is of the form

$x = A\cos\sqrt{\mu}\, t + B\sin\sqrt{\mu}\, t$ ……..(2)

where A and B are arbitrary constants.

Other forms of the solution equivalent to (2) are

$x = C\cos(\sqrt{\mu}\, t + \varepsilon)$…. (3) and $x = D\sin(\sqrt{\mu}\, t + \alpha)$ ………(4)

- If the solution of the S.H.M. equation is $x = a\cos(\sqrt{\mu}\, t + \varepsilon)$, the quantity ε is called the **epoch**.

Definition

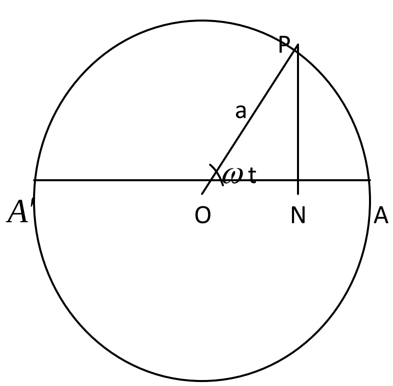
If two simple harmonic motions of the same period can be represented by

$x_1 = a_1\cos(\sqrt{\mu}\, t + \varepsilon_1)$ and $x_2 = a_2\cos(\sqrt{\mu}\, t + \varepsilon_2)$

- The difference in phase = $\frac{\varepsilon_1 - \varepsilon_2}{\sqrt{\mu}}$
- If $\varepsilon_1 = \varepsilon_2$ the motions are in the same phase.
- If $\varepsilon_1 = \varepsilon_2 = \pi$, they are in opposite phase.

4.2 Geometrical Representation of S.H.M

If a particle describes a circle with constant angular velocity, the foot of the perpendicular from the particle on a diameter moves with S.H.M.



Let AA' be the diameter of the circle with centre O and P be the position of the particle at time $t\,\text{secs}$. Let N be the foot of the perpendicular drawn from P on the diameter AA'. P moves along the circumference of the circle with uniform speed and describes equal arcs in equal times. Let ω – be the angular velocity. $\therefore \angle AOP = \omega t$

If ON = x, Op = a, then, x = a cos (ω t) ……………… (1)

$$\frac{dx}{dt} = -a\omega \sin(\omega t) \quad \text{………………..(2)}$$

$$\frac{d^2x}{dt^2} = -a\omega^2 \cos(\omega t) = -\omega^2 x \quad \text{……………… (3)}$$

(3) shows that the motion of N is simple harmonic. When P moves along the circumference of the circle starting from A, N oscillates from A to A' and A' to A.

Periodic time of P = Periodic time of N $= \dfrac{2\pi}{\omega}$

(along the circle) (along the diameter)

Problem 1

A particle is moving with S.H.M. and while making an oscillation from one extreme position to the other, its distances from the centre of oscillation at 3 consecutive seconds are $x_1, x_2, x_3.$ Prove that the period of oscillation is $\dfrac{2\pi}{\cos^{-1}\left(\dfrac{x_1 + x_3}{2x_2}\right)}$

Solution:

If a is the amplitude, μ the constant of the S.H.M. and x is the displacement at time t, we know that x = a cos $\sqrt{\mu}$ t ….. (1)

Let $x_1, x_2, x_3.$ be the displacements at three consecutive seconds $t_1,\ t_1 + 1,\ t_1 + 2$.

Then x_1 = a cos $\sqrt{\mu}$ t_1 ..…. (2)

x_2 = a cos $\sqrt{\mu}(t_1 + 1)$ = a cos $\left(\sqrt{\mu}t_1 + \sqrt{\mu}\right)$ …….(3)

x_3 = a cos $\sqrt{\mu}(t_1 + 2)$ = a cos $\left(\sqrt{\mu}t_1 + 2\sqrt{\mu}\right)$ …….(4)

$$\therefore x_1 + x_3 = a\,[\cos\left(\sqrt{\mu}\,t_1 + 2\sqrt{\mu}\right) + \cos\left(\sqrt{\mu}\,t_1\right)]$$

$$= a.2\cos\frac{\sqrt{\mu}t_1 + 2\sqrt{\mu} + \sqrt{\mu}t_1}{2}\ .\cos\ \frac{\sqrt{\mu}t_1 + 2\sqrt{\mu} - \sqrt{\mu}t_1}{2}$$

$$= 2\,a\cos\left(\sqrt{\mu}\ t_1 + \sqrt{\mu}\right).\cos\sqrt{\mu}\ = 2x_2\,.\cos\ \sqrt{\mu}$$

$$\therefore \frac{x_1 + x_3}{2x_2} = \cos\ \sqrt{\mu}\ , \qquad \sqrt{\mu} = \cos^{-1}\left(\frac{x_1 + x_3}{2x_2}\right)$$

$$\text{Period} = \frac{2\pi}{\sqrt{\mu}} = \frac{2\pi}{\cos^{-1}\left(\frac{x_1 + x_3}{2x_2}\right)}$$

Problem 2

If the displacement of a moving point at any time be given by an equation of the form $x = a\cos\ \omega\ t + b\sin\ \omega\ t$, show that the motion is a simple harmonic motion.

If a = 3, b=4, $\omega = 2$ determine the period, amplitude, maximum velocity and maximum acceleration of the motion.

Solution:

Given $x = a\cos\ \omega t + b\ \sin\varpi\ t$ ……………… (1)

Differentiating (1) with respect to t,

$$\frac{dx}{dt} = -a\omega\sin\ \omega t + b\,\omega\cos\omega t \quad \text{……………………(2)}$$

$$\frac{d^2x}{dt^2} = -\omega^2\ \cos\ \omega t - b\ \omega^2\sin\ \omega\ t$$

$$= -\omega^2(a\cos\ \omega\ t + b\sin\ \omega\ t) = -\omega^2 x \quad \text{….(3)}$$

$\therefore$ The motion is simple harmonic.

The constant μ of the S,H.M. = ω^2 .

$$\therefore \text{Period} = \frac{2\pi}{\sqrt{\mu}} = \frac{2\pi}{\omega} = \frac{2\pi}{2} = \pi \ \text{ secs.}$$

Amplitude is the greatest value of *x*.

When x is maximum, $\frac{dx}{dt} = 0$.

$-a\omega \sin \omega t + b\omega \cos \omega t = 0$ i.e. a sin ωt = b cos ωt or tan ω t = $\frac{b}{a} = \frac{4}{3}$

When tan ωt = $\frac{4}{3}$, sin ωt = $\frac{4}{5}$ and cos ωt = $\frac{3}{5}$

Greatest value of x = a $\times \frac{3}{5} + b \times \frac{4}{5} = \frac{3a+4b}{5} = \frac{3.3+4.4}{5} = 5$

Hence amplitude = 5.

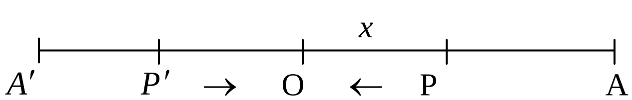
Max. acceleration = μ . Amplitude = 4 x 5 = 20

Max. velocity = $\sqrt{\mu}$. Amplitude = 2 x 5 =10

Problem 3

Show that the energy of a system executing S.H.M. is proportional to the square of the amplitude and of the frequency.

Solution:



The acceleration at a distance x from O = μx.

Force = mass $\times$acceleration = m μx

If the particle is given displacement dx from P,

work done against the force = m μ x. dx

Total work done in displacing the particle to a distance x

$$= \int_0^x m \ \mu x dx = m \ \mu \frac{x^2}{2} \qquad \text{………(1)}$$

Work done = potential energy at P.

If v is the velocity at P. we know that v $^2 = \mu\left(a^2 - x^2\right)$,

$\therefore$ Kinetic energy at P = $\frac{1}{2}$ mv ^{2} = $\frac{1}{2} m\mu\left(a^2 - x^2\right)$ ……….. (2)

The total energy at P = Potential energy + Kinetic energy

$$= \frac{m\mu x^2}{2} + \frac{m\mu}{2}\left(a^2 - x^2\right) = \frac{m\mu a^2}{2} \text{ ……...... (3)}$$

Total energy at P α a^2

If n is the frequency, we know that

$$n = \frac{1}{Period} = \frac{1}{\left(\frac{2\pi}{\sqrt{\mu}}\right)} = \frac{\sqrt{\mu}}{2\pi}$$

$\therefore$ $\sqrt{\mu} = 2\pi$ n or $\mu = 4\pi^2 n^2$

Total energy = $\frac{1}{2}m.\ 4\pi^2 n^2 a^2 = 2\pi^2 m a^2 n^2$ α n^2

Problem 4

A mass of 1 gm. Vibrates through a millimeter on each side of the midpoint of its path 256 times per sec; if the motion be simple harmonic, find the maximum velocity,

Solution:

Maximum velocity v = $\sqrt{\mu}$. a

Given, frequency $= \frac{1}{T}$ $= 256 = \frac{\sqrt{\mu}}{2\pi}$.

$\therefore$ $\sqrt{\mu}$ $= 2 \times 256 \times \pi$.

Given, amplitude = a = 1 millimeter = 1×10^{-1} c.m.

$\therefore$ Maximum velocity, V = $2 \times 256 \times \pi \times \frac{1}{10} = \frac{256\ \pi}{5}$ cm/ sec

Problem 5

In a S.H.M. if f be the acceleration and v the velocity at any time and T is the periodic time. Prove that $f^2T^2 + 4\pi^2v^2$ is constant.

Solution:

Velocity at any time, v $= \sqrt{\mu\left(a^2 - x^2\right)}$

Periodic time $\quad T = \dfrac{2\pi}{\sqrt{\mu}}, \ \dfrac{d^2x}{dt^2} = f.$

For, S.H.M, $\dfrac{d^2x}{dt^2} = -\mu.x$

$$\therefore f = -\mu.x$$

$$\therefore f^2T^2 + 4\pi^2v^2 = \mu^2x^2.\frac{4\pi^2}{\mu} + 4\pi\ \mu^2\left(a^2 - x^2\right)$$

$$= 4\pi^2\mu x^2 + 4\pi^2\mu a^2 - 4\pi^2\mu x^2$$

$$= 4\pi^2\mu a^2 \text{ (constant)}$$

Problem 6

A body moving with simple harmonic motion has an amplitude 'a' and period T. Show that the velocity v at a distance x from the mean position is given by $v^2T^2 = 4\pi^2\left(a^2 - x^2\right)$

Solution:

We know, $v^2 = \mu\left(a^2 - x^2\right)$

$$T = \frac{2\pi}{\sqrt{\mu}} \Rightarrow \mu = \frac{4\pi^2}{T^2}$$

$$\therefore v^2 = \frac{4\pi^2}{T^2}\left(a^2 - x^2\right)$$

$$\therefore v^2T^2 = 4\pi^2\left(a^2 - x^2\right)$$

Problem 7

If the amplitude of a S.H.M. is 'a' and the greatest speed is u, find the period of an oscillation and the acceleration at a given distance from the centre of oscillatin.

Solution:

Given, amplitude = a

Max. velocity = u.

ie) $\sqrt{\mu}\ a = u \Rightarrow \sqrt{\mu} = \dfrac{u}{a}$

Period of oscillation T = $\frac{2\pi}{\sqrt{\mu}}$ = $\frac{2\pi.a}{u}$ secs.

Acceleration $\frac{d^2x}{dt^2} = \mu\ x$ = $\frac{u^2 x}{a^2}$ units.

Problem 8

A particle, moving in S.H.M. has amplitude 8 cm. If its maximum acceleration is 2cm/sec ^{2}, find (i) its period (ii) maximum velocity and (iii) its velocity when it is 3 cm. from the extreme position

Solution:

Maximum acceleration = 2 cm/ sec ^{2} = $\mu.a. = \mu \times 8$.

$$\therefore \mu = \frac{2}{8} = \frac{1}{4},$$

Period T = $\frac{2\pi}{\sqrt{\mu}}$ = $2\pi \times \frac{1}{\sqrt{\frac{1}{4}}} = 4\pi$ secs.

Max. velocity = $\sqrt{\mu}.$ a = $\frac{1}{2} \times 8 = 4 cm/\sec.$

When the particle is 3 cm from the extreme position, x = 5 cm.

$$\therefore \text{ velocity}^2 = v^2 = \mu\left(a^2 - x^2\right) = \frac{1}{4}(64 - 25) = \frac{39}{4}.$$

$$\therefore v = \tfrac{1}{2}\sqrt{39} \text{ cm / sec.}$$

Problem 9

A particle moves in a straight line. If v be its velocity when at a distance x from a fixed point in the line and $v^2 = \alpha - \beta\ x^2$ where α, β are constants, show that the motion is simple harmonic and determinc its period and amplitude.

Solution:

Given, $v^2 = \alpha - \beta x^2$ ……………(1)

Differentiating, $2v.\ \frac{dv}{dt} = -2\beta x \frac{dx}{dt}\left[\because v = \frac{dx}{dt}\right]$

$\therefore\ \frac{dv}{dt} = -\beta x$

ie) $\boxed{\frac{d^2x}{dt^2} = -\beta\ x}$

$\therefore$ The motion is a S.H.M. $\sqrt{\mu}\ = \sqrt{\beta}$

Period T = $\frac{2\pi}{\sqrt{\mu}}\ = \frac{2\pi}{\sqrt{\beta}}$.

Amplitude is the maximum value of *x*.

x - is maximum, when $\frac{dx}{dt} = 0$

$\therefore v^2 = \alpha - \beta x^2\ = 0, \Rightarrow x = \sqrt{\frac{\alpha}{\beta}}$

$\therefore$ Amplitude = $\sqrt{\frac{\alpha}{\beta}}$

Problem 10

If the distance *x* of a point moving on a straight line measured from a fixed origin on it and velocity *v* are connected by the relation $4v^2 = 25 - x^2$, show that the motion is simple harmonic. Find the period and amplitude of the motion.

Solution:

Given, $4v^2 = 25\ - x^2$..............(1)

Differentiating, $8v.\ \frac{dv}{dt} = -2x.\frac{dx}{dt}$

$\therefore \frac{dv}{dt} = -\frac{1}{4}.x.$

$\boxed{\frac{d^2x}{dt^2} = -\frac{1}{4}.x.}$

Hence the motion is a S.H.M. Here $\mu = \frac{1}{4}$

$\therefore$ Period = $\dfrac{2\pi}{\sqrt{\mu}} = 2\pi\sqrt{4} = 4\pi$ secs.

Amplitude = maximum value of x.

x is maximum when $\dfrac{dx}{dt} = 0$

Ie) $25 - x^2 = 0. \Rightarrow x = \pm 5.$ Maximum value of x= 5.

$\boxed{\text{amplitude} = 5}$

4.3 Composition of two simple Harmonic Motions of the same period and in the same straight line

Since the period same, the two separate simple harmonic motions are represented by the same differential equation $\dfrac{d^2x}{dt^2} = -\mu\ x$

Let x_1 and x_2 be the displacements for the separate motions.

$x_1 = a_1 \cos\left(\sqrt{\mu}\ t + \varepsilon_1\right)$, a_1 - amplitude

$x_2 = a_2 \cos\left(\sqrt{\mu}\ t + \varepsilon_2\right)$, a_2 – amplitude

Let x be their resultant displacement, then $x = x_1 + x_2$

ie) $x = a_1 \cos\left(\sqrt{\mu} t + \varepsilon_1\right) + a_2 \cos\left(\sqrt{\mu} t + \varepsilon_2\right)$

$= a_1\left[\cos\sqrt{\mu} t.\cos\varepsilon_1 - \sin\sqrt{\mu} t.\sin\varepsilon_1\right] + a_2\left[\cos\sqrt{\mu} t.\cos\varepsilon_2 - \sin\sqrt{\mu} t.\sin\varepsilon_2\right]$

$= \cos\sqrt{\mu} t\left(a_1\cos\varepsilon_1 + a_2\cos\varepsilon_2\right) - \sin\sqrt{\mu} t\left(a_1\sin\varepsilon_1 + a_2\sin\varepsilon_2\right)$

$= \cos\sqrt{\mu} t.A\cos\varepsilon - \sin\sqrt{\mu} t.A\sin\varepsilon$ ……………….. (1)

where $A\cos\varepsilon = a_1\cos\varepsilon_1 + a_2\cos\varepsilon_2$ ……………..(2)

$A\sin\varepsilon = a_1\sin\varepsilon_1 + a_2\sin\varepsilon_2$ …………… (3)

Squaring (2) and (3) and adding,

$A^2 = a_1^2 + a_2^2 + 2a_1a_2\cos\left(\varepsilon_1 - \varepsilon_2\right)$ ………..(4)

Dividing (3) by (2), tan ε = $\dfrac{a_1 \sin\varepsilon_1 + a_2 \sin\varepsilon_2}{a_1 \cos\varepsilon_1 + a_2 \cos\varepsilon_2}$ ……..(5)

Now (1) becomes x = A . $\left(\cos\sqrt{\mu}t\cos\varepsilon - \sin\sqrt{\mu}t\sin\varepsilon\right)$

= A . cos $\left(\sqrt{\mu}t + \varepsilon\right)$ ……… (6)

The resultant displacement given by (6) also represents a simple harmonic motion of the same period as the individual motions.

4.4 Composition of two simple Harmonic motions of the same period in two perpendicular directions

If a particle possesses two simple harmonic motions of the same period, in two perpendicular directions, we can prove that its path is an ellipse. Take, two $\perp$r lines as x and y axes. The displacements of the particle can be taken as $x = a_1 \cos\sqrt{\mu}\ t$ …….. (1)

$$y = a_2 \cos\left(\sqrt{\mu}\ t + \varepsilon\right) \quad \text{……… (2)}$$

Eliminate ‘t’ between (1) and (2)

(2) $\Rightarrow$ $y = a_2 \cos\sqrt{\mu}\ t.\ \cos\ \varepsilon. - a_2 \sin\sqrt{\mu}\ t.\ \sin\ \varepsilon$

$$= a_2\left[\cos\ \varepsilon.\frac{x}{a_1} - \sin\ \varepsilon.\sqrt{1-\frac{x^2}{a_1^2}}\right] \text{ by (1)}$$

$$\frac{y}{a_2} = \cos\varepsilon.\frac{x}{a_1} - \sin\varepsilon.\sqrt{1-\frac{x^2}{a_1{}^2}}$$

i.e. $\dfrac{y}{a_2} - \dfrac{x\cos\varepsilon}{a_1} = -Sin\varepsilon.\sqrt{1-\dfrac{x^2}{a_1^2}}$

Squaring,

$$\frac{y^2}{a_2^2} + \frac{x^2\cos^2\varepsilon}{a_1^2} - \frac{2xy\cos\varepsilon}{a_1 a_2} = \sin^2\varepsilon - \frac{x^2}{a_1^2}\sin^2\varepsilon$$

i.e. $\frac{x^2}{a_1^2}-\frac{2xy}{a_1a_2}\cos\varepsilon+\frac{y^2}{a_2^2}=\sin^2\varepsilon$ ………. (3)

This is of the form $ax^2 + 2hxy + by^2 = \lambda$ ……. (4)

where $a = \frac{1}{a_1^{\,2}}$, $h = -\frac{\cos\ \varepsilon}{a_1a_2}$, $b = \frac{1}{a_2^{\,2}}$

(4) represents a conic with centre at the origin.

Also, $ab - h^2 = \frac{1}{a_1^{\,2}a_2^{\,2}} - \frac{\cos^2\ \varepsilon}{a_1^{\,2}a_2^{\,2}} = \frac{\sin^2\varepsilon}{a_1^{\,2}a_2^{\,2}} = +ve$

Hence (3) represents an ellipse.

If $\varepsilon = 0$, equation (3) $\Rightarrow \frac{x}{a_1}-\frac{y}{a_2}=0$ (straight line).

If $\varepsilon = \pi$, (3) $\Rightarrow \frac{x}{a_1}+\frac{y}{a_2}=0$ (straight line).

If $\varepsilon = \frac{\pi}{2}$, (3) $\Rightarrow \frac{x^2}{a_1^{\,2}}+\frac{y^2}{a_2^{\,2}}=1$ (ellipse).

If $\varepsilon = \frac{\pi}{2}$ and $a_1 = a_2$, the path is the circle $x^2+y^2 = a_1^{\,2}$

Problem 11

Show that the resultant of two simple harmonic motions in the same direction and of equal periodic time, the amplitude of one being twice that of the other and its phase a quarter of a period in advance, is a simple harmonic motion of amplitude $\sqrt{5}$ times that of the first and whose phase is in advance of the first by $\frac{\tan^{-1}2}{2\pi}$ of a period.

Solution:

Let the two displacements be

$x_1 = a_1\cos\left(\sqrt{\mu}\ \ t+\varepsilon_1\right)$…………(1) [$\because$ they have equal periodic time, μ is same]

$x_2 = a_2\cos\left(\sqrt{\mu}\ \ t+\varepsilon_2\right)$ ………… (2)

Where $a_2 = 2a_1$ and $\dfrac{\varepsilon_2 - \varepsilon_1}{\sqrt{\mu}}$ = phase difference (given) = $\dfrac{1}{4} \times \dfrac{2\pi}{\sqrt{\mu}}$

$\therefore \varepsilon_2 - \varepsilon_1 = \dfrac{\pi}{2}$ or $\varepsilon_2 = \dfrac{\pi}{2} + \varepsilon_1$

We know that the resultant displacement is x = A cos $\left(\sqrt{\mu}\ t + \varepsilon\right)$….. (3)

where $A^2 = a_1^2 + a_2^2 + 2a_1 a_2 \cos(\varepsilon_1 - \varepsilon_2)$

$= a_1^2 + 4a_1^2 + 4a_1^2 \cos\left(-90^0\right) = 5a_1^2$

$\therefore$ amplitude of the resultant motion = A = $a_1\sqrt{5}$

Also tan $\varepsilon = \dfrac{a_1 \sin\varepsilon_1 + a_2 \sin\varepsilon_2}{a_1 \cos\varepsilon_1 + a_2 \cos\varepsilon_2}$

[$\because A\sin\varepsilon = a_1 \sin\varepsilon_1 + a_2 \sin\varepsilon_2, \quad A\cos\varepsilon = a_1\cos\varepsilon_1 + a_2\cos\varepsilon_2$]

$= \dfrac{a_1 \sin\varepsilon_1 + 2a_1 \sin\left(90^0 + \varepsilon_1\right)}{a_1\cos\varepsilon_1 + 2a_1\cos\left(90^0 + \varepsilon_1\right)}$

i.e. $\dfrac{\sin\varepsilon}{\cos\varepsilon} = \dfrac{\sin\varepsilon_1 + 2\cos\varepsilon_1}{\cos\varepsilon_1 - 2\sin\varepsilon_1}$

$\sin\ \varepsilon\cos\varepsilon_1 - 2\sin\varepsilon\sin\varepsilon_1 = \sin\varepsilon_1\cos\varepsilon + 2\cos\varepsilon_1\ \cos\varepsilon$

$\therefore$ sin $(\varepsilon - \varepsilon_1) = 2\cos(\varepsilon - \varepsilon_1)$ i.e. tan $(\varepsilon - \varepsilon_1) = 2\ \therefore \varepsilon - \varepsilon_1 = \tan^{-1} 2$

$$\therefore \frac{\varepsilon - \varepsilon_1}{\sqrt{\mu}} = \frac{\tan^{-1} 2}{\sqrt{\mu}} = \frac{\tan^{-1} 2}{2\pi} \quad \left(\frac{2\pi}{\sqrt{\mu}}\right)$$

$= \dfrac{\tan^{-1} 2}{2\pi}$ of a period

Problem 12

Two simple harmonic motions in the same straight line of equal periods and differing in phase by $\dfrac{\pi}{2}$ are impressed simultaneously on a particle. If the amplitudes are 4 and 6, find the amplitude and phase of the resulting motion

Solution:

Let the two S.H.M. in the same straight line of equal periods and differing in phase by $\frac{\pi}{2}$ be,

$x_1 = a_1.\cos\sqrt{\mu}$ t ....(2)

$x_2 = a_2\left(\cos\sqrt{\mu}\ \ t + \varepsilon\right)$........(2)

given, A cos ε = 4 = a_1 , A sin ε = 6 = a_2

$\therefore$ Amplitude of the resultant motion A $= \sqrt{(A\cos\varepsilon)^2 + (ASin\varepsilon)^2}$

$= \sqrt{16+36} \quad = \sqrt{52}$

$$\boxed{A = 2\sqrt{13}}$$

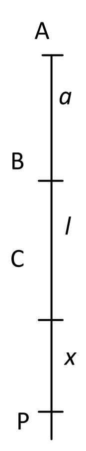
$$\tan\ \varepsilon = \frac{A\ \ Sin\varepsilon}{A\ \ Cos\varepsilon} = \frac{6}{4} = \frac{3}{2}$$

$$\therefore \quad \boxed{\varepsilon = \tan^{-1}\left(\frac{3}{2}\right)}$$

which is the phase of the resulting motion.

4.5 Motion of a particle suspended by a spiral spring

A particle is suspended from a fixed point by a spiral spring of natural length a and modulus λ. If it is displaced slightly in the vertical direction, discuss the subsequent motions



Let AB = a, natural length of the spring which is fixed at A. Let m be the mass of the particle connected at B, which pulls the spring and comes to rest at C such that the increased length BC = *l*. At C, the mass 'm' is in equilibrium. Hence the downward force mg and the upward force T must be equal at C. ie) T = mg

But, by Hooke's law, T = $\frac{\lambda l}{a}$

$$\therefore \frac{\lambda l}{a} = mg \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

Let the particle be slightly displaced vertically downwards through a distance and then released. It will begin to move upwards. Let P be the subsequent position of the particle so that CP= x

The forces acting at P are the weight and the upward tension.

Hence the equation of motion is

$$m\frac{d^2x}{dt^2} = \text{Resultant downward force} = mg - \text{Tension at P.}$$

$$= mg - \frac{\lambda}{a}(AP-AB)$$

$$= mg - \frac{\lambda}{a}(BP) = mg - \frac{\lambda}{a}(l + x)$$

$$= -\frac{\lambda x}{a} \quad \left[\because mg = \frac{\lambda l}{a}\right] \quad \text{by (1)}$$

$$\text{i.e. } \frac{d^2x}{dt^2} = -\frac{\lambda}{a\ m}x \quad \ldots.. (2)$$

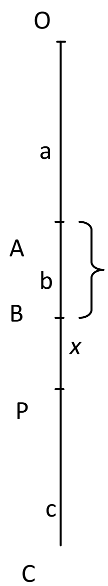
Equation (2) represents a S.H.M.

$$\text{Period} = \frac{2\pi}{\sqrt{\frac{\lambda}{am}}} = 2\pi\sqrt{\frac{am}{\lambda}}$$

Problem 13

Two bodies, of masses M and M', are attached to the lower end of an elastic string whose upper end is fixed and hang at rest; M' falls off. Show that the distance of M from the upper end of the string at time t is a+b+ c cos $\sqrt{\frac{g}{b}}$ t, where a is the unstretched length of the string, and b and c are the distances by which it would be stretched when supporting M and M', respectively.

Solution



Let OA = a be the natural length of the elastic string, which is fixed at O. When the string supports M,

Mg = upward Tension.

By Hooke's law,

upward Tension at B = $\frac{\lambda b}{a}$

$\therefore Mg = \frac{\lambda b}{a}$ ……………….. (1)

When the string supports M^1,

M^1g = upward Tension at C = $\frac{\lambda c}{a}$

ie) M^1g = $\frac{\lambda c}{a}$ …………………….. (2)

(1) + (2) $\Rightarrow M + M' = \frac{\lambda}{a}(b+c)$

ie) At C, M + M' is in equilibrium.

When M' falls off, M will move towards B.

Let P be the position of M at time t seconds such that BP = x

Forces acting at P are,

(i) Weight Mg ii) Upward tension

$\therefore$ At P, equation of motion of M is $M.\frac{d^2x}{dt^2}$ = resultant downward force.

$$= Mg - \frac{\lambda}{a}(OP - OA)$$

$$= Mg - \frac{\lambda}{a}(AP)$$

$$= Mg - \frac{\lambda}{a}(b + x)$$

$$= Mg - \frac{\lambda b}{a} - \frac{\lambda}{a}x$$

$$= -\frac{\lambda}{a}x \qquad \text{by (1)}$$

$$\boxed{\therefore \frac{d^2x}{dt^2} = -\frac{\lambda}{aM}.\ x}$$

$\therefore$ The motion of M at P is simple harmonic

Amplitude = BC = c

$\therefore$ Displacement = $x = c.\cos\sqrt{\frac{\lambda}{aM}}\ t$

$$= c.\cos\sqrt{\frac{g}{b}}.\ t \qquad \text{by (1)}$$

$\therefore$ Distance of M from O at time t = OP = OA + AB + BP

= a + b +x

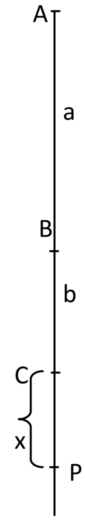
= a + b + c. cos $\sqrt{\frac{g}{b}}.\ t$

Problem 14

A Particle of mass m is tied to one end of an elastic string which is suspended from the other end. The extension caused in its length is b. If the particle is pulled down and let go, show that it executes simple harmonic motion and that the period is $2\pi\sqrt{\dfrac{b}{g}}$

Solution:

Let AB be the natural length of the elastic string. When m is tied at the other end, extended length is b. and the mass is in equilibrium at C.



$\therefore$ At C, mg = T = $\dfrac{\lambda b}{a}$ ____________(1)

When the mass is pulled down and released let P be the subsequent position such that CP = x

At P, equation of motion is

m. $\dfrac{d^2x}{dt^2}$ = resultant downward force

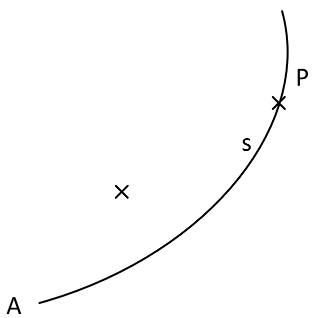
= mg $-\dfrac{\lambda(b+x)}{a} = -\dfrac{\lambda x}{a}$ [by (1)]

$$\boxed{\therefore \quad \frac{d^2x}{dt^2} = -\frac{\lambda}{am}.x. \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_(2)}$$

(2) shows that the motion is simple harmonic

$\therefore$ Period T = $\dfrac{2\pi}{\sqrt{\mu}} = \dfrac{2\pi}{\sqrt{\dfrac{\lambda}{am}}} = 2\pi\sqrt{\dfrac{am}{\lambda}} = 2\pi\sqrt{\dfrac{b}{g}}$ by (1)

4.6 Simple Harmonic Motion On a Curve



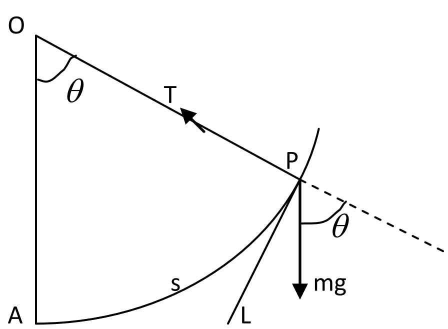
If P is the position of a particle on a curve at time t and if the tangential acceleration at P varies as the arcual distance of P measured from a fixed point A on the curve and is directed towards A, then the motion of P is said to be simple harmonic.

The differential equation for the S.H.M. on a curve will be of the form $\frac{d^2s}{dt^2} = -\mu s$, s is the arc distance AP.

4.7 Simple pendulum

A simple pendulum consists of a small heavy particle or bob suspended from a fixed point by means of a light inextensible string and oscillating in a vertical plane.

Period of oscillation of a simple pendulum



Let OA = l be the length of the pendulum where O is the point of suspension. Let 'm' be the mass of the bob and P be the position of the bob in time t secs and arc AP = s, $A\hat{O}P = \theta$

The two forces acting are i) mg$(\downarrow)$ ii) Tension T along PO.

mg is resolved into two components i) mg $\cos\theta$ along OP.

ii) mg $\sin\theta$ along PL.

mg $\cos\theta$ and T balances each other.

The equation of motion at P is $m.\dfrac{d^2s}{dt^2} = -mg.\sin\theta$ …………… (1)

[Negative sign shows that mg $\sin\theta$ is towards A.]

When θ is small, sin $\theta \cong \theta$

$$\therefore \frac{d^2s}{dt^2} = -g.\theta \qquad \text{………………………… (2)}$$

But s = $l\theta,\ \theta = \dfrac{s}{l},\ \therefore \dfrac{d^2s}{dt^2} = -\dfrac{g}{l}.s$ ………………………… (3)

(3) shows that the motion of the bob at P is **simple harmonic when θ is small.**

Hence $\mu = \dfrac{g}{l}$

$$\text{Period T} = \frac{2\pi}{\sqrt{\mu}} = \frac{2\pi}{\sqrt{\frac{g}{l}}} = 2\pi\sqrt{\frac{l}{g}}$$

4.8 Simple equivalent pendulum

A simple pendulum which oscillates in the same time as the given pendulum is called the Simple Equivalent Pendulum.

Consider two motions represented by the equations.

$$\frac{d^2x}{dt^2} = -\mu x \text{ …… (1)}$$

$$\frac{d^2s}{dt^2} = -\frac{g}{l}s \text{ …….(2)}$$

We know that (1) and (2) are S.H. motions and (2) is the equation of motion of a simple pendulum.

They represent equivalent motions, if $\mu = \dfrac{g}{l}$ i.e. $\boxed{l = \dfrac{g}{\mu}}$

The length of the simple equivalent pendulum is $\dfrac{g}{\mu}$.

4.9 The Seconds Pendulum

A seconds pendulum is one whose period of oscillation is 2 seconds.

Hence if l is its length, we have $2 = 2\pi\sqrt{\frac{l}{g}}$ $\therefore$ $l = \frac{g}{\pi^2}$

The length of the seconds pendulum is $\frac{g}{\pi^2}$

Note : Since the time of oscillation of a seconds pendulum is 2 secs, it makes 43200 oscillation per day. If it gains n seconds a day, it makes $43200 + \frac{n}{2}$ oscillations in 86,400 secs.

Hence its period = $\dfrac{86400}{43200 + \frac{n}{2}}$ ……………………. (1)

If it loses n seconds a day, it makes $43200 - \frac{n}{2}$ oscillation in 864000 secs.

So its period = $\dfrac{86400}{43200 - \frac{n}{2}}$ …………………………(2)

Problem 15

Find the length of a simple pendulum which oscillates 56 times in 55 seconds

Solution:

Given, T = $\frac{55}{56}$ secs.

But T = $2\pi\sqrt{\frac{l}{g}}$ l- length of the pendulum

$$\therefore \; 2\pi\sqrt{\frac{l}{g}} = \frac{55}{56}$$

$$\therefore \; \sqrt{\frac{l}{g}} = \frac{55}{56\times 2\pi} = \frac{55\times 7}{56\times 2\times 22} = \frac{5}{32}$$

$$\therefore \; \frac{l}{g} = \left(\frac{5}{32}\right)^2 = \frac{25}{1024}$$

$$\therefore l = \frac{25}{1024}\times 9.8 = 0.239\,m.$$

Problem 16

Show that an incorrect seconds pendulum of a clock which loses x seconds a day must be shortened by $\frac{x}{432}$ percent of its length in order to keep correct time.

Solution:

Let l, l^1 be the correct and incorrect lengths of the seconds pendulum of a clock

$$\therefore \mathrm{T} = 2\pi\sqrt{\frac{l}{g}} = \frac{86400}{43200} = 2 \text{ secs} \_\_\_\_\_\_\_\_\_(1)$$

When it loses x seconds a day,

$$2\pi\sqrt{\frac{l^1}{g}} = \frac{86400}{43200 - \frac{x}{2}} \qquad \_\_\_\_\_\_\_\_(2)$$

$$\frac{(2)}{(1)} \Rightarrow \sqrt{\frac{l^1}{l}} = \frac{43200}{43200 - \frac{x}{2}} = \frac{1}{1 - \frac{x}{86400}}$$

$$\therefore \frac{l^1}{l} = \frac{1}{\left(1 - \frac{x}{86400}\right)^2} = \left(1 - \frac{x}{86400}\right)^{-2} = 1 + \frac{2x}{86400} \text{ (approximately)}$$

ie) $\frac{l^1}{l} = 1 + \frac{x}{43200}$

$$\therefore l^1 = l + \frac{x}{43200} l$$

ie) $l^1 = l + \frac{x}{432}$ Percent of l

$\therefore$ Length should be shortened by $\frac{x}{432}$ percent of its length in order to keep correct time.

Problem 17

A pendulum whose length is l makes m oscillations in 24 hours. When its length is slightly altered, it makes m+n oscillations in 24 hours. Show that the diminution of the length is $\frac{2nl}{m}$ nearly**.**

Solution:

Given , when the length of the pendulum is *l*, it makes 'm' oscillations in 24 hrs.

$$\therefore \; T = 2\pi\sqrt{\frac{l}{g}} = \frac{24}{m} \qquad \_\_\_\_\_\_\_\_\_\_\_(1)$$

When its length is altered, let $l - l^1$ be its length and it makes m+n oscillations per day.

$$\therefore \text{ Periodic time } T = 2\pi\sqrt{\frac{l-l^1}{g}} = \frac{24}{m+n} \qquad \_\_\_\_\_\_\_\_\_\_\_(2)$$

$$\therefore \; \frac{(1)}{(2)} \Rightarrow \frac{m+n}{m} = \sqrt{\frac{l}{l-l^1}}$$

$$\text{ie)} \; \sqrt{\frac{1}{1-\frac{l^1}{l}}} = 1 + \frac{n}{m}$$

$$\text{ie)} \; \left(1 - \frac{l^1}{l}\right)^{-1/2} = 1 + \frac{n}{m}$$

$$\text{ie)} \; 1 + \frac{l^1}{2l} = 1 + \frac{n}{m} \text{ (nearly)}$$

$$\therefore l' = \frac{2nl}{m} \text{ nearly}$$

Problem 18

A seconds pendulum which gains 10 seconds per day at one place loses 10 seconds per day at another. Compare the acceleration due to gravity at the two places.

Solution:

Let g_1 , g_2 be the acceleration due to gravity at the two places where the pendulum gains 10 secs per day and loses 10 secs per day respectively.

When it gains, Periodic time = $2\pi\sqrt{\dfrac{l}{g_1}} = \dfrac{24\times 60\times 60}{43200+5}$ ______(1)

When it loses, Periodic time = $2\pi\sqrt{\dfrac{l}{g_2}} = \dfrac{24\times 60\times 60}{43200-5}$ _______(2)

where l is the length of the pendulum

$$\therefore \quad \frac{(1)}{(2)} \Rightarrow \sqrt{\frac{g_2}{g_1}} = \frac{43195}{43205} \quad \therefore \frac{g_1}{g_2} = \frac{(43205)^2}{(43195)^2}$$

Problem 19

If l_1 is the length of an imperfectly adjusted seconds pendulum which gains n seconds in one hour and l_2 the length of one which loses n seconds in one hour at the same place, show that the true length of the seconds pendulum is $\dfrac{4\ l_1\ l_2}{l_1+l_2+2\sqrt{l_1 l_2}}$

Solution:

Let l be the true length of the seconds pendulum. For the same place g is constant,

$$\therefore \ T = 2\pi\sqrt{\frac{l}{g}} = 2 \text{ secs} \qquad \text{\_\_\_\_\_\_\_ (1)}$$

Let l_1 be the length of the pendulum, when it gains n seconds in one hour.

$$\therefore \text{Period} = 2\pi\sqrt{\frac{l_1}{g}} = \frac{3600}{1800+\dfrac{n}{2}} \text{\_\_\_\_\_\_\_\_\_\_\_\_(2)}$$

Let l_2 - be the length of the pendulum, when it loses n seconds in one hour.

$$\therefore \text{ Period } \ 2\pi\sqrt{\frac{l_2}{g}} = \frac{3600}{1800-\dfrac{n}{2}} \qquad \text{\_\_\_\_\_\_\_\_\_(3)}$$

$$\frac{(1)}{(2)} \Rightarrow \sqrt{\frac{l}{l_1}} = \frac{1800 + \frac{n}{2}}{1800} = 1 + \frac{n}{3600} \quad \text{\_\_\_\_\_\_\_\_\_\_\_\_(4)}$$

$$\frac{(1)}{(3)} \Rightarrow \sqrt{\frac{l}{l_2}} = \frac{1800 - \frac{n}{2}}{1800} = 1 - \frac{n}{3600} \quad \text{\_\_\_\_\_\_\_\_\_(5)}$$

(4) +(5) $\Rightarrow \sqrt{\frac{l}{l_1}} + \sqrt{\frac{l}{l_2}} = 2$

Squaring, $\frac{l}{l_1} + \frac{l}{l_2} + \frac{2l}{\sqrt{l_1 l_2}} = 4$

$$\therefore\ l\left(\frac{l_2 + l_1}{l_1 l_2} + \frac{2}{\sqrt{l_1 l_2}}\right) = 4$$

i.e.) $l\left(\frac{l_1 + l_2 + 2\sqrt{l_1 l_2}}{l_1 l_2}\right) = 4$

$$\boxed{\therefore\ l = \frac{4 l_1 l_2}{l_1 + l_2 + 2\sqrt{l_1 l_2}}}$$

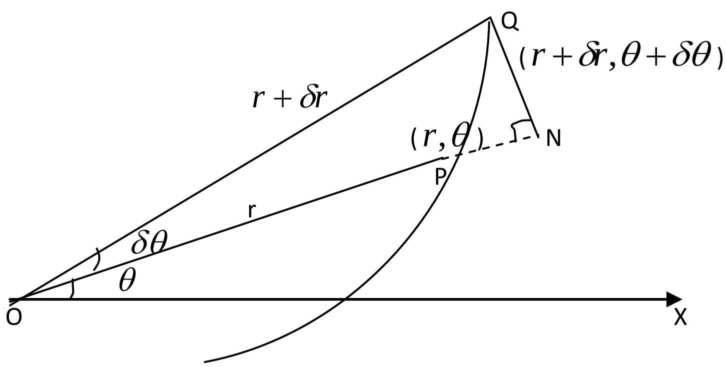
UNIT V

MOTION UNDER THE ACTION OF CENTRAL FORCES

In this unit we study components of velocities and accelerations in two mutually perpendicular directions. We deal with the motion under the action of a force always directed towards a fixed point and derive formulae for various velocities and accelerations together with polar form and pedal form of central orbits.

5.1 Velocity and acceleration in polar co- ordinates

Radial and Transverse velocities



Consider a particle moves in a plane curve. Let P (r, θ) be its position in time t and $Q(r+\delta r, \theta+\delta\theta)$ be its position in time t+δt. Take O – as the pole and OX- as initial line. Velocity along the radius vector OP in the direction of r increasing is called the radial velocity and the velocity in the direction $\perp$ r to OP in the direction of θ increasing is called the transverse velocity.

Radial velocity at P = $\lim_{\delta t \to 0} \left[\dfrac{\text{displacement along OP in time } \delta t}{\delta t} \right]$

$$= \lim_{\delta t \to 0} \frac{PN}{\delta t} = \lim_{\delta t \to 0} \frac{ON - OP}{\delta t}$$

$$= \lim_{\delta t \to 0} \frac{(r+\delta r)\cos\delta\theta - r}{\delta t}$$

$$= \lim_{\delta t \to 0} \frac{(r+\delta r)\left[1+\frac{(\delta\theta)^2}{2!}+\ldots\ldots\right]-r}{\delta t}$$

$$= \lim_{\delta t \to 0} \frac{(r+\delta t)(1)-r}{\delta t}$$, neglecting higher powers of $\delta\theta$

$$= \lim_{\delta t \to 0} \frac{\delta r}{\delta t} = \frac{dr}{dt} = \dot{r}$$

$\therefore$ $\boxed{\text{Radial velocity } = \dot{r}}$

Transverse velocity at P = $\lim_{\delta t \to 0} \frac{QN}{\delta t} = \lim_{\delta t \to 0} \frac{(r+\delta r).\sin\delta\theta}{\delta t}$

$$= \lim_{\delta t \to 0} \frac{(r+\delta r)\left[\delta\theta - \frac{(\delta\theta)^3}{3!}+\ldots\ldots\ldots\right]}{\delta t}$$

$$= \lim_{\delta t \to 0} \frac{(r+\delta r)\delta\theta}{\delta t}$$, neglecting higher powers of $\delta\theta$

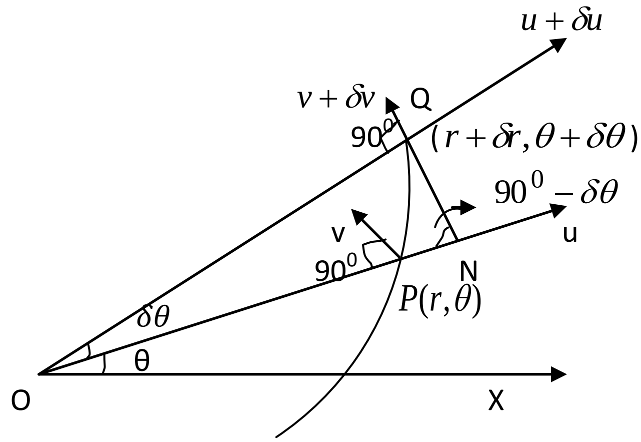
$$= \lim_{\delta t \to 0} \frac{r\ \delta\theta}{\delta t} = \left(r.\lim_{\delta t \to 0} \frac{\delta\theta}{\delta t}\right)$$

$$= r\frac{d\theta}{dt} = r\ \dot{\theta}$$

Transverse velocity = r $\dot{\theta}$

Radial and Transverse Accelerations

Let u, v be the radial and transverse velocities at (r,θ) and $(u+\delta u)$ and $(v+\delta v)$ be the radial and transverse velocities at Q $(r+\delta r,\theta+\delta\theta)$



Radial acceleration = $\lim_{\delta t \to 0} \left[\dfrac{Change\ \ of\ \ velocity\ along\ OP\ in\ time\ \delta t}{\delta t} \right]$

$$= \lim_{\delta t \to 0} \left[\frac{(u+\delta u)\cos\delta\theta - (v+\delta v)\cos(90^\circ - \delta\theta)}{\delta t} \right] - u$$

$$= \lim_{\delta t \to 0} \frac{[(u+\delta u)[1] - (v+\delta v)(\delta\theta) - u]}{\delta t}$$

$$= \lim_{\delta t \to 0} \frac{\delta u - v\delta\theta}{\delta t}$$

$$= \lim_{\delta t \to 0} \frac{\delta u}{\delta t} - v \lim_{\delta t \to 0} \frac{\delta\theta}{\delta t}$$

$$= \frac{du}{dt} - v\frac{d\theta}{dt}, \text{ where u } = \frac{dr}{dt}, v = r\frac{d\theta}{dt}$$

$$= \frac{d}{dt}\left(\frac{dr}{dt}\right) - r\frac{d\theta}{dt}.\frac{d\theta}{dt}$$

$$= \frac{d^2 r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 = \ddot{r} - r\dot{\theta}^2$$

$$\boxed{\therefore Radial\ \ acceleration = \ddot{r} - r\dot{\theta}^2}$$

Transverse acceleration = $\underset{\delta t \to 0}{Lim} \dfrac{\left[Change\ in\ velocity\ \ perpendicular\ \ to\ \ OP\ \ \text{int}\ ime\ \ \delta t\right]}{\delta t}$

$$= \underset{\delta t \to 0}{Lim} \frac{\left[(u+\delta u)\sin\delta\theta + (v+\delta v)\sin\left(90^\circ - \delta\theta\right)\right] - v}{\delta t}$$

$$= \underset{\delta t \to 0}{Lim} \frac{\left[(u+\delta u)\sin\delta\theta + (v+\delta v)\cos\delta\theta\right] - v}{\delta t}$$

when $\delta\theta$ is small, $\sin\delta\theta \approx \delta\theta$

$$= \underset{\delta t \to 0}{Lim} \frac{\left[(u+\delta u)(\delta\theta) + (v+\delta v)(1) - v\right]}{\delta t} \quad \text{and} \quad \cos\delta\theta \approx 1$$

$$= \underset{\delta t \to 0}{Lim} \left[\frac{u\delta\theta + \delta v}{\delta t}\right] = u\frac{d\theta}{dt} + \frac{dv}{dt} \text{ Where u} = \frac{dr}{dt}, v = r\frac{d\theta}{dt}$$

$$= \frac{dr}{dt}.\frac{d\theta}{dt} + \frac{d}{dt}\left(r\frac{d\theta}{dt}\right)$$

$$= \frac{dr}{dt}.\frac{d\theta}{dt} + r\frac{d^2\theta}{dt^2} + \frac{d\theta}{dt}.\frac{dr}{dt}$$

$$= \text{r}\ \frac{d^2\theta}{dt^2} + 2\frac{dr}{dt}.\frac{d\theta}{dt}.$$

$$= \frac{1}{r}\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right) = \frac{1}{r}\frac{d}{dt}\left(r^2\dot{\theta}\right)$$

$$\boxed{\therefore Transverse\ \ acceleration = \frac{1}{r}\frac{d}{dt}\left(r^2\dot{\theta}\right)}$$

		Magnitude
1	Radial Component of velocity	$\dot{r}$
2	Transverse Component of velocity	$r\dot{\theta}$
3	Radial component of acceleration	$\ddot{r} - r\dot{\theta}^2$
4	Transverse component of acceleration	$\frac{1}{r}\frac{d}{dt}\left(r^2\dot{\theta}\right)$

Corollary

(1) Suppose the particle P is describing a circle of radius 'a'. Then r = a throughout the motion

Hence $\ddot{r} = 0$ and the **radial acceleration =** $\ddot{r} - r\dot{\theta}^2$

$$= 0 - a\dot{\theta}^2 = -a\dot{\theta}^2$$

Transverse acceleration = $\frac{1}{r} \cdot \frac{d}{dt}(r^2\dot{\theta}) = \frac{1}{a}a^2\ddot{\theta} = a\ddot{\theta}$

(2) The magnitude of the resultant velocity of P

$$= \sqrt{\dot{r}^2 + (r\dot{\theta})^2} = \sqrt{\dot{r}^2 + r^2\dot{\theta}^2)}$$

and the magnitude of the resultant acceleration

$$= \sqrt{(\ddot{r} - r\dot{\theta}^2)^2 + [\frac{1}{r} \cdot \frac{d}{dt}(r^2\dot{\theta})]^2}$$

Problem 1

The velocities of a particle along and perpendicular to a radius vector from a fixed origin are λr^2 and $\mu\theta^2$ where μ and λ are constants. Show that the equation to the path of the particle is $\frac{\lambda}{\theta} + C = \frac{\mu}{2r^2}$ where C is a constant. Show also that the accelerations along and perpendicular to the radius vector are $2\lambda^2 r^3 - \frac{\mu^2\theta^4}{r}$ *and* $\mu\left(\lambda r\theta^2 + \frac{2\mu\theta^3}{r}\right)$

Solution:

Radial velocity = $\frac{dr}{dt} = \lambda r^2$ …. (1)

Transverse velocity = $r\frac{d\theta}{dt} = \mu\theta^2$ ….. (2)

Dividing (2) by (1), we have

$$r\frac{d\theta}{dr}=\frac{\mu\theta^2}{\lambda r^2} \qquad i.e.\lambda\frac{d\theta}{\theta^2}=\frac{\mu}{r^3}dr$$

Integrating, $-\frac{\lambda}{\theta}=-\frac{\mu}{2r^2}+C$

i.e. $\frac{\mu}{2r^2}=\frac{\lambda}{\theta}+C$ …. (3)

(3) is the equation of the path,

Differentiating (1) $\frac{d^2r}{dt^2}=\lambda\cdot 2r\frac{dr}{dt} = 2\lambda^2r^3$ using (1)

Radial acceleration = $\ddot{r}-r\dot{\theta}^2=\frac{d^2r}{dt^2}-r(\frac{d\theta}{dt})^2$

$$= 2\lambda^2r^3-r(\frac{\mu\theta^2}{r})^2=2\lambda^2r^3-\frac{\mu^2\theta^4}{r} \text{ using (2)}$$

Transverse acceleration $=\frac{1}{r}\cdot\frac{d}{dt}(r^2\dot{\theta})=\frac{1}{r}\cdot\frac{d}{dt}(r^2\frac{\mu\theta^2}{r})$

$$=\frac{1}{r}\cdot\frac{d}{dt}(\mu\ r\theta^2)=\frac{\mu}{r}(r^2\theta\frac{d\theta}{dt}+\theta^2\frac{dr}{dt})$$

$$=\frac{\mu}{r}(2r\cdot\theta\frac{\mu\theta^2}{r}+\theta^2\cdot\lambda r^2)=\mu[\frac{2\mu\theta^3}{r}+\lambda r\theta^2]$$

Problem 2

The velocities of a particle along and perpendicular to the radius from a fixed origin are λ r and $\mu\ \theta$; find the path and show that the acceleration along and perpendicular to the radius vector are $\lambda^2 r-\frac{\mu^2\theta^2}{r}$ and $\mu\ \theta\left(\lambda+\frac{\mu}{r}\right)$

Solution:

Given, radial velocity = $\dot{r}=\frac{dr}{dt}=\lambda r$ __________(1)

Transverse velocity = r$\dot{\theta}=\mu\theta$ __________(2)

Radial acceleration $=\ddot{r}-r\dot{\theta}^2$

$$= \lambda\ \dot{r} - r\left(\frac{\mu\theta}{r}\right)^2 \qquad \text{[by (1) \& (2)]}$$

$$= \lambda(\lambda\ r) - \frac{\mu^2\theta^2}{r} = \lambda^2 r - \frac{\mu^2\theta^2}{r}$$

Transverse acceleration $= \frac{1}{r}\frac{d}{dt}\left(r^2\dot{\theta}\right) = \frac{1}{r}\frac{d}{dt}\left(r^2.\frac{\mu\theta}{r}\right)$

$$= \frac{1}{r}\frac{d}{dt}\left(\mu\ r\ \theta\right) = \frac{1}{r}.\mu\left[r\dot{\theta} + \theta\ \dot{r}\right]$$

$$= \frac{\mu}{r}\left[r.\frac{\mu\theta}{r} + \theta.\lambda\ r\right]$$

$$\boxed{\text{Transverse ace.} = \mu\theta\left[\frac{\mu}{r} + \lambda\right]}$$

$$\frac{(2)}{(1)} \Rightarrow \frac{r\frac{d\theta}{dt}}{\frac{dr}{dt}} = \frac{\mu\theta}{\lambda\ r} = \frac{\mu}{\lambda}.\frac{\theta}{r}.$$

i.e. $\frac{r\ d\theta}{dr} = \frac{\mu}{\lambda}.\frac{\theta}{r} \quad \therefore \frac{d\theta}{\theta} = \frac{\mu}{\lambda}.\frac{dr}{r^2}$

Integrating, $\log\theta = \frac{\mu}{\lambda}\left[\frac{r^{-1}}{-1}\right] + C;$ C – constant

$$= -\frac{\mu}{\lambda\ r} + C$$

i.e. $\boxed{\log\quad \theta = c - \frac{\mu}{\lambda r}}$

which is the equation of the path

Problem 3

The velocities of a particle along and perpendicular to the radius vector from a fixed origin area and b. Find the path and the acceleration along and perpendicular to the radius vector.

Solution:

Radial velocity = $\dot{r} = \frac{dr}{dt} = a$ ________(1)

Transverse velocity = $r\dot{\theta} = r\frac{d\theta}{dt} = b$ ______(2)

Radial acceleration = $\ddot{r} = r\dot{\theta}^2 = \frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2$

Now, $\ddot{r} = \frac{d}{dt}(\dot{r}) = \frac{da}{dt} = 0$

$\therefore$ Radial acceleration $= -r\left(\frac{b}{r}\right)^2 = -\frac{b^2}{r}$

Transverse acceleration $= \frac{1}{r}.\frac{d}{dt}\left(r^2\dot{\theta}\right) = \frac{1}{r}.\frac{d}{dt}\left(r^2.\frac{b}{r}\right)$

$= \frac{1}{r}.\frac{d}{dt}(br) \quad = \frac{b}{r}.\frac{dr}{dt} = \frac{ab}{r}$

To find the path

$\frac{(2)}{(1)} \Rightarrow \frac{r\frac{d\theta}{dt}}{\frac{dr}{dt}} = \frac{b}{a}$ i.e. $r\frac{d\theta}{dr} = \frac{b}{a} \Rightarrow \frac{dr}{r} = \frac{a}{b}d\theta$

Integrating, log r = $\frac{a}{b}\theta + c,$ where C – is constant

$\boxed{\therefore r = A.e^{\frac{a\theta}{b}}}$ is the equation of the path.

Problem 4

A point moves so that its radial and transverse velocities are always 2 λ a θ and λr. Show that its accelerations in these two directions are λ^2(2a-r) and that its path is the curve r = a $\theta^2 + C$.

Solution:

Given, radial velocity $\dot{r} = \dfrac{dr}{dt} = 2\lambda\,\mathrm{a}\,\theta$ ______________ (1)

Transverse velocity r $\dot{\theta}$ = r$\dfrac{d\theta}{dt} = \lambda r$ ___________ (2) $\Rightarrow \dot{\theta} = \lambda$

Radial acceleration (R.A) = $\ddot{r} - r\dot{\theta}^2 = 2\,\lambda a\dfrac{d\theta}{dt} - r\left[\lambda^2\right]$

$$= 2\ \lambda a \cdot \lambda - r\lambda^2$$

$$\boxed{\mathrm{R} \cdot A = \lambda^2(2a - r)}$$

Transverse acceleration (T.A) = $\dfrac{1}{r}\dfrac{d}{dt}\left(r^2\dot{\theta}\right)$

$$= \frac{1}{r}\frac{d}{dt}\left(r^2 \cdot \lambda\right) = \frac{\lambda}{r} \cdot 2r\ \dot{r}$$

$$= \frac{\lambda}{r} .\ 2\mathrm{r}\ .\ 2\lambda a\theta$$

$$\boxed{\mathrm{T.A} = 4\ \lambda^2 a\theta}$$

$$\frac{(2)}{(1)} \Rightarrow \frac{r\dfrac{d\theta}{dt}}{\dfrac{dr}{dt}} = \frac{\lambda\ r}{2\lambda a\theta} = \frac{r}{2a\theta} \text{ i.e. r } \frac{d\theta}{dr} = \frac{r}{2a\theta}$$

$$\therefore 2a\ \theta d\theta = dr$$

Integrating, 2a $\dfrac{\theta^2}{2} + C = r, C$ - constant

$\boxed{\mathrm{r} = \mathrm{C} + \mathrm{a}\theta^2}$ is the equation of the path.

Problem 5

If a point moves so that its radial velocity is k times its transverse velocity then show that its path is an equiangular spiral.

Solution:

Given, radial velocity = $k \times$ transverse velocity

i.e. $\dot{r} = k.r\dot{\theta}$

i.e. $\frac{dr}{dt} = k.r.\frac{d\theta}{dt}$

$$\therefore \frac{dr}{r} = k.d\theta$$

Integrating, log r = k θ + log A, A – constant

i.e. $\log\left(\frac{r}{A}\right) = k\theta \quad \therefore \frac{r}{A} = e^{k\theta}$

$$\boxed{r = A\ e^{k\theta}}$$

which is an equiangular spiral.

Problem 6

If the radial and transverse velocities of a particle are always proportional to each other, show that the equation of the path is of the form r = A. $e^{k\theta}$, where A and k are constants.

Solution:

Given radial velocity α transverse velocity

i.e $\dot{r}\ \alpha\ r\dot{\theta} \Rightarrow \dot{r} = k.r\dot{\theta},$ k – constant

$$\Rightarrow \frac{dr}{r} = k.d\theta$$

Integrating, log r = K$\theta +$ log A

log r – log A = k θ

ie) log $\left(\frac{r}{A}\right) = k\theta$

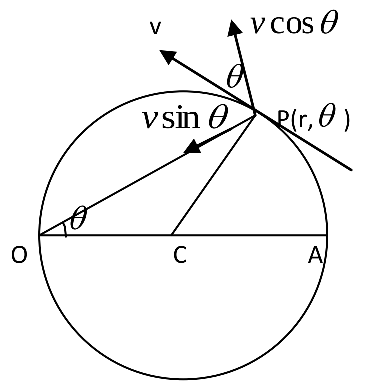
i.e $\log\left(\frac{r}{A}\right) = k..\theta$

$$\Rightarrow \frac{r}{A} = e^{k\theta} \Rightarrow \quad \boxed{r = A.e^{k\theta}}$$

Problem 7

A point moves in a circular path of radius 'a' so that its angular velocity about a fixed point in the circumference of the circle is constant, equal to ω.Show that the resultant acceleration of the point at every point of the path is of constant magnitude 4 aω^2.

Solution:



Let O – be the fixed point (pole), OC – initial line. Polar equation of the circle is r = 2 a cos θ. Let P (r, θ) be the position at time't' Angular velocity about O is $\dot{\theta} = w$ (constant)

Radial velocity $= \dot{r} = -(2a\sin\theta)\dot{\theta} = -2a\omega\sin\theta$

$$\ddot{r} = -(2a\omega\cos\theta)\dot{\theta} = -2a\omega^2\cos\theta$$

$$= -\omega^2(2a\cos\theta)$$

$$= -\omega^2.r$$

Radial acceleration at P $= \ddot{r} - r\dot{\theta}^2$

$= -\omega^2 r - r.\omega^2$

$= -2\omega^2 r = -2\omega^2(2a\cos\theta)$

$= -4a\omega^2\cos\theta$

Transverse acceleration at P $= \frac{1}{r}.\frac{d(r^2\dot{\theta})}{dt} = \frac{1}{r}.w.2r\,\dot{r}$

$$= 2\ \omega(-2a\omega\sin\theta) = -4a\omega^2\sin\theta$$

$\therefore$ Resultant acceleration $= \sqrt{(-4a\omega^2\cos\theta)^2 + (-4a\omega^2\sin\theta)^2}$

$$= 4a\ \omega^2$$

Problem 8

A point moves with uniform speed v along a cardioid r = a (1+ cosθ). Show that (i) its angular velocity ω about the pole is v$\dfrac{\sec\theta/2}{2a}$ (ii) the radial component of the acceleration is constant equal to $\dfrac{3v^2}{4a}$ (iii) the magnitude of the resultant acceleration is $\dfrac{3v\omega}{2}$.

Solution:

Given, path is r = a $(1+\cos\theta)$ ………………(1)

Uniform speed v = resultant velocity = $\sqrt{\dot{r}^2 + (r\dot{\theta})^2}$

(1) $\Rightarrow \dot{r} = a(-\sin\theta)\dot{\theta}$

$$\ddot{r} = -a[\sin\theta.\ddot{\theta} + \dot{\theta}\cos\theta.\dot{\theta}]$$

$$= -(a\cos\theta)\dot{\theta}^2 - a\ddot{\theta}\sin\theta.$$

$\therefore$ v $= \sqrt{a^2\dot{\theta}^2\sin^2\theta + [a(1+\cos\theta)\dot{\theta}]^2}$

$$= \sqrt{a^2\dot{\theta}^2\sin^2\theta + a^2\dot{\theta}^2(1+2\cos\theta+\cos^2\theta)}$$

$$= \sqrt{a^2\dot{\theta}^2 + a^2\dot{\theta}^2(1+2\cos\theta)}$$

$$= \sqrt{2a^2\dot{\theta}^2 + 2\cos\theta.a^2\dot{\theta}^2}$$

$$= a\ \dot{\theta}\sqrt{2}\quad\sqrt{1+\cos\theta}$$

$$= \sqrt{2}\ a\ \dot{\theta}\sqrt{2\cos^2\theta/2}$$

$$\boxed{v = 2\,a\,\dot{\theta}.\cos\theta/2}$$

$$\therefore \dot{\theta} = \omega = \frac{v}{2a.\cos\theta/2} = \left(\frac{v}{2a}\right).\sec\theta/2$$

Radial acceleration $= \ddot{r} - r\dot{\theta}^2$

$$= -(a\cos\theta)\dot{\theta}^2 - (a\sin\theta)\ddot{\theta} \quad - a(1+\cos\theta)\dot{\theta}^2$$

$$= -a(1+2\cos\theta)\dot{\theta}^2 - \dot{\theta}(a\sin\theta)\,{}^{1}\!/_{2}\left[\frac{v}{2a}\sec{}^{\theta}\!/_{2}.\tan{}^{\theta}\!/_{2}\right]$$

$$= -a(1+2\cos\theta)\dot{\theta}^2 - \frac{a}{2}\sin\theta.\tan{}^{\theta}\!/_{2}\left[\frac{v}{2a}.\sec{}^{\theta}\!/_{2}\right]\dot{\theta}$$

$$= -a(1+2\cos\theta)\left[\frac{v}{2a}.\sec{}^{\theta}\!/_{2}\right]^2 - \frac{a}{2}\sin\theta\tan{}^{\theta}\!/_{2}\left[\frac{v}{2a}\sec.{}^{\theta}\!/_{2}\right]^2$$

$$= -a\left(\frac{v^2}{4a^2}\right)\left(\sec^2{}^{\theta}\!/_{2}\right)\left[(1+2\cos\theta) + \frac{1}{2}\tan{}^{\theta}\!/_{2}.\sin\theta\right]$$

$$= -\frac{1}{4}\left(\frac{v^2}{a}\right)\left(\sec^2{}^{\theta}\!/_{2}\right)\left[(1+2\cos\theta) + \sin^2\frac{\theta}{2}\right]$$

$$= -\frac{1}{4}\left(\frac{v^2}{a}\right)\left(\sec^2{}^{\theta}\!/_{2}\right)\left[(1+2\cos\theta) + \frac{1}{2}(1-\cos\theta)\right]$$

$$= -\frac{1}{4}\left(\frac{v^2}{a}\right)\left(\sec^2{}^{\theta}\!/_{2}\right)\left[\frac{3}{2}(1+\cos\theta)\right]$$

$$= -\frac{3}{4}\left(\frac{v^2}{a}\right)\left(\sec^2{}^{\theta}\!/_{2}\right)[(1+\cos\theta)]$$

$$= -\frac{3}{4}\left(\frac{v^2}{a}\right)\sec^2\frac{\theta}{2}.\cos^2\frac{\theta}{2} = -\frac{3}{4}\left(\frac{v^2}{a}\right)$$

$$\boxed{\text{R.A = constant}}$$

Transverse acceleration $= \dfrac{1}{r}.\dfrac{d}{dt}\left(r^2\dot{\theta}\right)$

$$= \frac{1}{r}\cdot\frac{d}{dt}\left[a^2(1+\cos\theta)^2\cdot\frac{v}{2a}\cdot\sec{}^{\theta}\!/_{2}\right]$$

$$= \frac{1}{r}\frac{d}{dt}\left[a^2.\left(2\cos^2{}^{\theta}\!/_{2}\right)^2\cdot\frac{v}{2a}\sec{}^{\theta}\!/_{2}\right]$$

$$= \frac{1}{r}\cdot\frac{d}{dt}\left[2a.\cos^3 \theta/2 .v\right]$$

$$= \frac{1}{r}\times 2a.v.\left[3\cos^2 \theta/2 \cdot\left(-\sin \theta/2\right)\dot{\theta}/2\right]$$

$$= -\frac{3av}{r}\cdot\cos^2 \theta/2 \cdot\left[\frac{v}{2a}\sec\theta/2\right] .\sin\frac{\theta}{2}$$

$$= -\frac{3v^2}{2r}\left[\cos\theta/2 \cdot\sin\theta/2\right]$$

$$= -\frac{3v^2}{2\left[a(1+\cos\theta)\right]}\times\left[\cos\theta/2 \cdot\sin\theta/2\right]$$

$$= -\frac{3v^2}{2a.2\cos^2 \theta/2}\cdot\cos\theta/2 \cdot\sin\theta/2$$

$$\boxed{\text{T.A} = -\frac{3v^2}{4a}\cdot\tan\theta/2}$$

$\therefore$ Resultant acceleration = $\pm\sqrt{(R.A)^2+(T.A)^2}$

$$= \pm\sqrt{\left(-\frac{3}{4}\frac{v^2}{a}\right)^2+\left(\frac{-3v^2}{4a}\tan\theta/2\right)^2}$$

$$= \pm\sqrt{\frac{9v^4}{16a^2}\left(1+\tan^2 \theta/2\right)}$$

$$= \pm\sqrt{\frac{9v^4}{16a^2}\cdot\sec^2 \theta/2}$$

$$= \pm\frac{3v^2}{4a}\sec\theta/2$$

$$= \pm\left[\frac{3v\omega}{2}\right]$$

5.2 Differential Equation of central orbits

A particle moves in a plane with an acceleration which is always directed to a fixed point O in the plane. Obtain the differential equation of its path.

Take O as the pole and a fixed line through O as the initial line. Let P (r, θ) be the polar coordinates of the particle at time t and m be its mass. Also let P be the magnitude of the central acceleration along PO.

The equations of motion of the particle are

m $\left(\ddot{r} - r\dot{\theta}^2\right)$ = - mP

i.e. $\ddot{r} - r\dot{\theta}^2$ = - P ……. (1)

and $\frac{m}{r} \cdot \frac{d}{dt}\left(r^2\dot{\theta}\right) = 0$

i.e. $\frac{1}{r} \cdot \frac{d}{dt}\left(r^2\dot{\theta}\right) = 0$ ……. (2)

Equation (2) shows that the transverse component of the acceleration is zero throughout the motion.

From (2), $r^2\dot{\theta}$ = constant = h ……… (3)

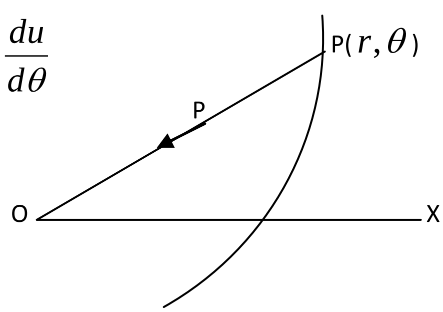
To get the polar equation of the path, we have to eliminate t between (1) and (3).

put u = $\frac{1}{r}$

From (3), $\dot{\theta} = \frac{h}{r^2} = h\ u^2$

Also $\dot{r} = \frac{dr}{dt} = \frac{d}{dt}\left(\frac{1}{u}\right) = -\frac{1}{u^2}\frac{du}{dt} = -\frac{1}{u^2}\frac{du}{dt} = -\frac{1}{u^2}\frac{du}{d\theta} \cdot \frac{d\theta}{dt}$

$$= -\frac{1}{u^2}\frac{du}{d\theta} \cdot h\ u^2 = -h\frac{du}{d\theta}$$



$$\ddot{r} = \frac{d}{dt}\left(-h\frac{du}{d\theta}\right) = -h\frac{d}{d\theta}\left(\frac{du}{d\theta}\right) \cdot \frac{d\theta}{dt}$$

$$= -h\frac{d^2u}{d\theta^2} \cdot hu^2 = -h^2u^2\frac{d^2u}{d\theta^2}$$

Substitute r and $\dot{\theta}$ in (1), we get

$$-h^2u^2\frac{d^2u}{d\theta}-\frac{1}{u}h^2u^4=-\text{P ie } h^2u^2\left(\frac{d^2u}{d\theta^2}+u\right)=\text{P}$$

ie) u + $\dfrac{d^2u}{d\theta^2}=\dfrac{P}{h^2u^2}$ …..(4)

(4) is the differential equation of a central orbit, in polar coordinates.

Perpendicular from the pole on the tangent - Formulae in polar coordinates

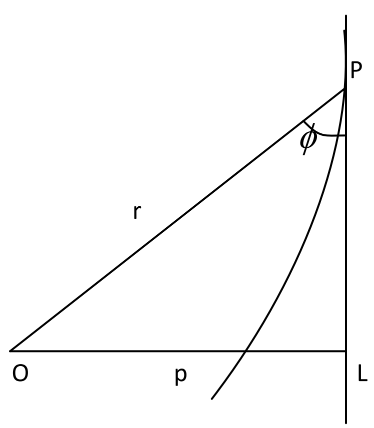
Let φ be the angle made by the tangent at P with the radius vector OP.

We know that $\tan\varphi = r\dfrac{d\theta}{dr}$ ……… (1)

From O draw OL perpendicular to the tangent at P and let OL= p.

Then sin φ = $\dfrac{OL}{OP}=\dfrac{p}{r}$

$\therefore$ p = r sin φ ……. (2)



Now eliminate φ between (1) and (2).

From (2), $\dfrac{1}{p^2}=\dfrac{1}{r^2\sin^2\varphi}=\dfrac{1}{r^2}\operatorname{cosec}^2\varphi$

$$=\frac{1}{r^2}\left(1+\cot^2\varphi\right)$$

$$=\frac{1}{r^2}\left[1+\frac{1}{r^2}\left(\frac{dr}{d\theta}\right)^2\right],\text{(by (1))}$$

i.e. $\dfrac{1}{p^2}=\dfrac{1}{r^2}+\dfrac{1}{r^4}\left(\dfrac{dr}{d\theta}\right)^2$ ….. (3)

Using r = $\frac{1}{u}, \frac{dr}{d\theta} = \frac{dr}{du} \cdot \frac{du}{d\theta} = -\frac{1}{u^2} \cdot \frac{du}{d\theta}$

Hence (3) becomes

$$\frac{1}{P^2} = u^2 + u^4 . \frac{1}{u^4}\left(\frac{du}{d\theta}\right)^2$$

i. e) $\frac{1}{p^2} = u^2 + \left(\frac{du}{d\theta}\right)^2$ ……………… (4)

5.3 Pedal equation (or) (p, r) equation of the central orbit

We have $\frac{1}{p^2} = u^2 + \left(\frac{du}{d\theta}\right)^2$ ……… (1)

Differentiating both sides of (1) with respect to θ,

$$-\frac{2}{p^3} \cdot \frac{dp}{d\theta} = 2u\frac{du}{d\theta} + 2\frac{du}{d\theta} \cdot \frac{d^2u}{d\theta^2} = 2\frac{du}{d\theta}\left(u + \frac{d^2u}{d\theta^2}\right) \quad …….. (2)$$

But the differential equation is u + $\frac{d^2u}{d\theta^2} = \frac{P}{h^2u^2}$

Hence (2) becomes $-\frac{1}{p^3} \cdot \frac{dp}{d\theta} = \frac{P}{h^2u^2} \cdot \frac{du}{d\theta}$

i.e. $-\frac{1}{p^3} dp = \frac{P}{h^2u^2} du = \frac{P}{h^2} r^2 d\left(\frac{1}{r}\right)$

$$= \frac{P\, r^2}{h^2} \times -\frac{1}{r^2} dr = -\frac{P}{h^2} dr$$

$$\boldsymbol{\frac{h^2}{p^3} \cdot \frac{dp}{dr} = P} \quad ………(3)$$

is the (p, r) equation or the pedal equation to the central orbit.

Problem 9

Find the law of force towards the pole under which the curve

$r^n = a^n . \cos n\theta$ can be described.

Solution:

Given $r^n = a^n \cos n\theta$

Put $r = \frac{1}{u}$, the equation is $u^n a^n \cos n\theta = 1$ …… (1)

Taking logarithms,

$n \log u + n \log a + \log \cos n\theta = 0$ ……. (2)

Differentiating (2) with respect to θ

$$n \frac{1}{u}\frac{du}{d\theta} - \frac{n \sin n\theta}{\cos n\theta} = 0$$

ie) $\frac{du}{d\theta} = u \tan n\theta$ …………….. (3)

Differentiating (3) with respect to θ,

$$\frac{d^2u}{d\theta^2} = un \sec^2 n\theta + \tan n\theta . \frac{du}{d\theta}$$

$$= nu \sec^2 n\theta + u \tan^2 n\theta \text{ using (3)}$$

$$u + \frac{d^2u}{d\theta} = u + nu \sec^2 n\theta + u \tan^2 n\theta$$

$$= nu \sec^2 n\theta + u\left(1 + \tan^2 n\theta\right)$$

$$= nu \sec^2 n\theta + u \sec^2 n\theta = (n+1)u \sec^2 n\theta$$

$$= (n+1)u.u^{2n}a^{2n} \text{ using (1)}$$

$$= (n+1)a^{2n}u^{2n+1}$$

$$P = h^2u^2\left(u + \frac{d^2u}{d\theta^2}\right) = h^2u^2.(n+1)a^{2n}u^{2n+1}$$

$$= (n+1)a^{2n}.h^2.u^{2n+3}$$

$$= (n+1)a^{2n}.h^2.\frac{1}{r^{2n+3}} \qquad \text{………..} (4)$$

$$\therefore \ P \ \alpha \frac{1}{r^{2n+3}}$$

Important notes

(i) When n=1, the equation is r = a cos θ. The curve is a circle and P $\alpha\ 1\ /r^5$.

(ii) When n = 2, the equation is $r^2 = a^2 \cos 2\theta$. This is the Lemniscate of Bernowli and P $\alpha\ \frac{1}{r^7}$.

(iii) When n = $\frac{1}{2}$, the equation is $r^{\frac{1}{2}} = a^{\frac{1}{2}} \cos\frac{\theta}{2}$

i.e. r = a $\cos^2\frac{\theta}{2} = \frac{a}{2}(1+\cos\theta)$

This is a cardioid and P $\alpha\ \frac{1}{r^4}$

(iv) When n = $-\frac{1}{2}$, the equation is $r^{\frac{-1}{2}} = a^{\frac{-1}{2}}.\cos\frac{\theta}{2}$

i.e. $a^{\frac{1}{2}} = r^{\frac{1}{2}} \cos\frac{\theta}{2}$

So r = $\frac{a}{\cos^2\frac{\theta}{2}} = \frac{2a}{1+\cos\theta}$ i.e $\frac{2a}{r} = 1+\cos\theta$

This is a parabola and P $\alpha\ \frac{1}{r^2}$

(v) When n = - 2 , the equation is $r^{-2} = a^{-2}.\cos 2\theta$

i.e. $r^2 \cos 2\theta = a^2$ (rectangular hyperbola)

Problem 10

A particle moves in an ellipse under a force which is always directed towards its focus. Find the law of force, the velocity at any point of the path and its periodic time.

Solution:

The polar equation to the ellipse, with pole at focus is

$$\frac{l}{r} = 1 + e\cos\theta \qquad \text{........ (1)}$$

where e is the eccentricity and *l* is the semi latus-rectum.

From (1), u = $\frac{1}{r} = \frac{1+e\ \cos\theta}{l}$

Hence $\frac{du}{d\theta} = -\frac{e\ \sin\theta}{l}$ and $\frac{d^2u}{d\theta^2} = -\frac{e\ \cos\theta}{l}$

u+ $\frac{d^2u}{d\theta^2} = \frac{1+e\ \cos\theta}{l} - \frac{e\ \cos\theta}{l} = \frac{1}{l}$

We know that $\frac{P}{h^2u^2} = u + \frac{d^2u}{d\theta^2} = \frac{1}{l}$

Hence P = $\frac{h^2u^2}{l} = \frac{\mu}{r^2}$, **where** $\mu = \frac{h^2}{l}$

i.e. ***The force varies inversely as the square of the distance from the pole.***

Now, $\frac{1}{p^2} = u^2 + \left(\frac{du}{d\theta}\right)^2$

$$= \left(\frac{1+e\ \cos\theta}{l}\right)^2 + \left(\frac{e\ \sin\theta}{l}\right)^2 = \frac{1+2e\ \cos\theta + e^2}{l^2}$$

Also h = pv where v is the linear velocity

$\therefore$ v $^2 = \frac{h^2}{p^2} = \frac{h^2\left(1+2e\ \cos\theta + e^2\right)}{l^2}$

$$= \frac{\mu l}{l^2}\left[1+e^2 + 2\left(\frac{l}{r} - 1\right)\right] \text{ from (1)}$$

$$= \frac{\mu}{l}\left(e^2 + \frac{2l}{r} - 1\right) = \frac{\mu}{l}\left[\frac{2l}{r} - \left(1-e^2\right)\right]$$

$$= \mu\left[\frac{2}{r} - \frac{\left(1-e^2\right)}{l}\right] \qquad \text{........ (2)}$$

Now a and b are the semi major and minor - axes of the ellipse.

$$\therefore\ l = \frac{b^2}{a} = \frac{a^2\left(1-e^2\right)}{a} = a\left(1-e^2\right)$$

Put $l = a\left(1-e^2\right)$ in (2)

$$\mathbf{v}^2 = \mu\left[\frac{2}{r} - \frac{1}{a}\right],\ \therefore V = \sqrt{\mu\left[\frac{2}{r} - \frac{1}{a}\right]}$$

Areal velocity = $\dfrac{h}{2}$

Area of the ellipse = πab

Periodic Time T $= \dfrac{\pi\ ab}{\left(\frac{h}{2}\right)} = \dfrac{2\pi\ ab}{h}$

$$= \frac{2\pi\ ab}{\sqrt{\mu l}} = \frac{2\pi\ ab}{\sqrt{\mu}.b}.\sqrt{a}$$

$$= \frac{2\pi}{\sqrt{\mu}}.a^{\frac{3}{2}}$$

Problem 11

Find the law of force towards the pole under which the curves can be described.

i) $r^2 = a^2 \cos 2\theta$

[Hint : Put n = 2 in problem 9, (i.e., $r^n = a^n.\cos n\theta$)]

ii) $r^{1/2} = a^{1/2} \cos \theta/2$

[Hint : Put n = $1/2$ in problem 9,(i.e.,$r^n = a^n \cos n\theta$)]

iii.) $r^n \cos n\theta = a^n$

Solution:

$a^n u^n = \cos n\theta \left[\because r = \frac{1}{u}\right]$ __________(1)

Take log both sides, and differentiate n log a + n log u = log cosn θ

$$\frac{n}{u}\cdot\frac{du}{d\theta} = \frac{1}{\cos n\theta}(-\sin n\theta)n$$

$$\therefore \frac{du}{d\theta} = -u\tan n\theta \quad \_\_\_\_\_\_\_\_\_(2)$$

$$\therefore \frac{d^2u}{d\theta^2} = -\left[u.\sec^2 n\theta.n + (\tan n\theta)\frac{du}{d\theta}\right]$$

$$= -\left[un\sec^2 n\theta - u.\tan^2 n\theta\right]$$

$$= u\tan^2 n\theta - un.\sec^2 n\theta$$

We have, u + $\frac{d^2u}{d\theta^2} = \frac{P}{h^2u^2}$

i.e., u+ u $\tan^2 n\theta - un.\sec^2 n\theta = \frac{P}{h^2u^2}$

u $\sec^2 n\theta(1-n) = \frac{p}{h^2u^2}$

i.e P= $h^2(1-n).u^3.\frac{1}{a^{2n}.u^{2n}} = \frac{h^2(1-n)}{a^{2n}}.\frac{1}{u^{2n-3}}$

$$= \frac{h^2(1-n)}{a^{2n}}.r^{2n-3}$$

$$\boxed{\therefore P \propto r^{2n-3}}$$

iv) $\mathbf{r}^n = A\cos n\theta + B.\sin n\theta$

Solution:

This equation can be taken as

$r^n = \lambda.\cos(n\theta+\alpha), \lambda, \alpha$ are constants.

$$\therefore 1 = \lambda . u^n . \cos(n\theta + \alpha), \because r = \frac{1}{u}$$

Take log both sides and differentiate,

$$0 = \log \lambda + n \log u + \log \cos (n\theta + \alpha)$$

$$\therefore n . \frac{1}{u} . \frac{du}{d\theta} + \frac{1}{\cos(n\theta + \alpha)} [-\sin(n\theta + \alpha)] \text{ n} = 0$$

$$\therefore \frac{du}{d\theta} = u . \tan(n\theta + \alpha) \quad \underline{\qquad\qquad}(1)$$

$$\therefore \frac{d^2u}{d\theta^2} = u . \sec^2(n\theta + \alpha) . n + \tan(n\theta + \alpha) . \frac{du}{d\theta}$$

$$= \text{nu. } \sec^2(n\theta + \alpha) + u . \tan^2(n\theta + \alpha)$$

$$= \text{n.u. } \sec^2(n\theta + \alpha) + u . \left[\sec^2(n\theta + \alpha) - 1\right]$$

$$\therefore u + \frac{d^2u}{d\theta^2} = (n+1) u \sec^2(n\theta + \alpha) = \frac{P}{h^2u^2}$$

$$\therefore P = h^2 . (n+1) . u^3 . \sec^2(n\theta + \alpha)$$

$$= \text{h}^2 (n+1) . u^3 \left(\lambda u^n\right)^2$$

$$= \text{h}^2 \lambda^2 (n+1) . u^{2n+3} = \frac{\lambda^2 h^2 (n+1)}{r^{2n+3}}$$

$$\therefore \qquad \boxed{P \quad \alpha \frac{1}{r^{2n+3}}}$$

v) a = r sin nθ

Solution:

Take log and differentiate au = $\sin n\theta$

$$\log (\text{au}) = \log \sin n \theta \qquad \left[\because r = \frac{1}{u}\right]$$

i.e. log a + log u = log (sin nθ)

$$\therefore \frac{1}{u}.\frac{du}{d\theta} = \frac{1}{\sin n\theta}.\cos n\theta.n$$

$$\boxed{\frac{du}{d\theta} = nu.\cot n\theta}$$

$$\therefore \frac{d^2u}{d\theta^2} = n\left[u.\left(-\operatorname{cosec}^2 n\theta\right)n + \cot n\theta.\frac{du}{d\theta}\right]$$

$$= \text{n}\left[-nu.\operatorname{cosec}^2 n\theta + nu.\cot^2 n\theta\right]$$

$$= \text{n}^2.\text{u}\left[\cot^2 n\theta - \operatorname{cosec}^2 n\theta\right] = -n^2u.$$

$$\therefore u + \frac{d^2u}{d\theta^2} = u - n^2u = u\left(1-n^2\right)$$

But, $\text{u} + \dfrac{d^2u}{d\theta^2} = \dfrac{P}{h^2u^2} = u\left(1-n^2\right)$

$$\therefore P = h^2u^3\left(1-n^2\right)$$

$$= \text{h}^2\left(1-n^2\right).u^3 = \frac{h^2\left(1-n^2\right)}{r^3}$$

$\therefore$

$$\boxed{\text{P} \ \alpha \frac{1}{r^3}}$$

vi) r = a sin nθ

Solution:

1 = au. sin nθ . $\qquad \left[\because r = \dfrac{1}{u}\right]$

Take log and differentiate,

0 = log a + log u + log sin n θ

$$\frac{1}{u}.\frac{du}{d\theta} + \frac{1}{\sin n\theta}\left(\cos n\theta\right).n = 0$$

i.e $\frac{1}{u}.\frac{du}{d\theta}+n.\cot n\theta=0$

$\therefore$ $\boxed{\frac{du}{d\theta}=-nu.\cot n\theta}$

$$\therefore \frac{d^2u}{d\theta^2}=-n\left[u.\left(-\cos ec^2 n\theta.n\right)+\cot n\theta.\frac{du}{d\theta}\right]$$

$$= -n\left[-nu.\cos ec^2 n\theta-nu\cot^2 n\theta\right]$$

$$=n^2u\left[\cos ec^2 n\theta+\cot^2 n\theta\right]$$

$$\therefore u+\frac{d^2u}{d\theta^2}=u+n^2u\cos ec^2 n\theta+n^2u\cot^2 n\theta$$

$$= \text{u}+n^2u.\frac{a^2}{r^2}+n^2u\left(\cos ec^2 n\theta-1\right)$$

$$= u+\frac{n^2a^2}{r^3}+n^2u.\frac{a^2}{r^2}-\frac{n^2}{r}$$

$$= \frac{1}{r}+2\frac{n^2a^2}{r^3}-\frac{n^2}{r} = \frac{2n^2a^2}{r^3}-\frac{\left(n^2-1\right)}{r}$$

$$\text{But } u+\frac{d^2u}{d\theta^2}=\frac{P}{h^2u^2}$$

$$\therefore \frac{2n^2a^2}{r^3}-\frac{\left(n^2-1\right)}{r}=\frac{P}{h^2u^2}$$

$$\therefore P=h^2\left[\frac{2n^2a^2}{r^5}-\frac{\left(n^2-1\right)}{r^2}\right]$$

$$\therefore P\alpha\left[\frac{2n^2a^2}{r^5}-\frac{\left(n^2-1\right)}{r^3}\right]$$

vii) $\frac{a}{r} = e^{n\theta}$

Solution:

Given $\frac{a}{r} = e^{n\theta}$

$\therefore au = e^{n\theta}$ __________(1) $\left[\because r = \frac{1}{u}\right]$

Differentiating, a.$\frac{du}{d\theta} = e^{n\theta} n$

$$\therefore \frac{du}{d\theta} = \frac{n}{a}.e^{n\theta}$$

$$\therefore \frac{d^2u}{d\theta^2} = \frac{n}{a}.e^{n\theta}.n$$

$$= \frac{n^2}{a}.e^{n\theta}$$

$$\therefore u + \frac{d^2u}{d\theta} = u + \frac{n^2}{a}.e^{n\theta} = \frac{e^{n\theta}}{a} + \frac{n^2}{a}e^{n\theta}$$

$$= \frac{e^{n\theta}}{a}\left(1+n^2\right) = u\left(1+n^2\right) \text{ by (1)}$$

But, $u + \frac{d^2u}{d\theta^2} = \frac{P}{h^2u^2} = u\left(1+n^2\right)$

$$\therefore P = h^2u^3\left(1+n^2\right) = h^2\left(1+n^2\right).u^3$$

$$= \frac{h^2\left(1+n^2\right)}{r^3}$$

$$\therefore \boxed{P \ \alpha \frac{1}{r^3}}$$

viii) r = a. e $^{\theta \cot \alpha}$

Solution:

Given r = a. e $^{\theta \cot \alpha}$

$$1 = \text{au. e}^{\theta \cot \alpha} \qquad \text{\_\_\_(1)} \qquad \left[\because u = \frac{1}{r}\right]$$

Differentiating w.r.to θ,

$$0 = a\left[u.e^{\theta \cot \alpha}.\cot \alpha + e^{\theta \cot \alpha}.\frac{du}{d\theta}\right]$$

$$\therefore \frac{du}{d\theta} = -\frac{u.e^{\theta \cot \alpha}.\cot \alpha}{e^{\theta \cot \alpha}} = -u \cot \alpha$$

$$\therefore \frac{d^2u}{d\theta^2} = -\cot \alpha \quad .\frac{du}{d\theta} = \text{u} \cot^2 \alpha$$

$$\therefore u + \frac{d^2u}{d\theta} = u + u\cot^2 \alpha = u(1 + \cot^2 \alpha) = u.\cos ec^2 \alpha$$

But u+ $\dfrac{d^2u}{d\theta^2} = \dfrac{P}{h^2u^2} = u.\cos ec^2 \alpha$

$$\therefore P = h^2u^3.\cos ec^2 \alpha$$

$$= \frac{h^2.\cos ec^2 \alpha}{r^3}$$

$$\boxed{\therefore P \alpha \frac{1}{r^3}}$$

ix) r = a cosh n θ

Solution:

$$1 = \text{au. cosh n } \theta \text{ \_\_\_\_\_\_\_\_\_\_(1)} \left[\because r = \frac{1}{u}\right]$$

Differentiating w.r.to θ, a $\left[u.n.\sinh n\theta + \cosh n\theta.\dfrac{du}{d\theta} \right] = 0$

$$\therefore \frac{du}{d\theta} = -nu \quad \tanh \quad n\theta \qquad \_\_\_\_\_\_\_\_\_\_(2)$$

$$\therefore \frac{d^2u}{d\theta^2} = -n\left[un\text{sec}h^2 n\theta + \tanh \quad n\theta.\frac{du}{d\theta} \right]$$

$$= -n\left[nu.\text{sec}h^2 n\theta - nu\tanh^2 n\theta \right]$$

$$= -n^2u\left[\text{sec}h^2 n\theta - \tanh^2 n\theta \right]$$

$$\therefore \frac{d^2u}{d\theta^2} = -n^2u\left[\text{sec}h^2 n\theta + \text{sec}h^2 n\theta - 1 \right] \left[\because \text{sec}h^2\theta + \tanh^2\theta = 1 \right]$$

$$= -n^2u\left[2\text{sec}h^2 n\theta - 1 \right]$$

$$= -n^2u\left[2a^2u^2 - 1 \right]$$

$$\therefore u + \frac{d^2u}{d\theta^2} = -2n^2a^2u^3 + n^2u + u \quad = -2n^2a^2u^3 + \left(n^2 + 1 \right) u.$$

But, u+ $\dfrac{d^2u}{d\theta^2} = \dfrac{P}{h^2u^2}$

$$\therefore \frac{P}{h^2u^2} = -2n^2a^2u^3 + \left(n^2 + 1 \right) u.$$

$$\therefore P = -2n^2a^2h^2.u^5 + h^2\left(n^2 + 1 \right) u^3$$

$$= -\frac{2n^2a^2h^2}{r^5} + \frac{\left(n^2 + 1 \right)h^2}{r^3}$$

$$\boxed{\text{P}\,\alpha - \frac{2n^2a^2}{r^5} + \frac{\left(n^2 + 1 \right)}{r^3}}$$

x) r cosh n θ = a

Solution:

Given r cosh n θ = a

$\therefore$ au = cosh nθ ………… (1) $\left[\because \frac{1}{r} = u\right]$

Differentiating w.r.to θ,

$$a. \frac{du}{d\theta} = n.\sinh n\theta$$

$$\therefore a.\frac{d^2u}{d\theta^2} = n^2.\cosh n\theta$$

$$\frac{d^2u}{d\theta^2} = \frac{n^2}{a}\cosh n\theta$$

But, $u + \frac{d^2u}{d\theta^2} = \frac{P}{h^2u^2}$

$$\therefore u + \frac{n^2}{a}.\cosh n\theta = \frac{P}{h^2u^2}$$

$$u + \frac{n^2}{a}.au = \frac{P}{h^2u^2} \qquad \text{[from (1)]}$$

i.e. $u + n^2u = \frac{P}{h^2u^2}$

$$\therefore P = h^2u^3(1 + n^2) = \frac{h^2(n^2 + 1)}{r^3}$$

$$\boxed{\therefore P\alpha \frac{1}{r^3}}$$

Problem 12

Find the central acceleration under which the conic $\frac{l}{r}$ = 1 + e cos θ, can be described.

Solution:

Given equation is, $lu = 1 + e\cos\theta$ $\qquad\qquad \because \frac{1}{r} = u$

$$\therefore u = \frac{1+e\cos\theta}{l} = \frac{1}{l} + \frac{e}{l}.\cos\theta$$

$$\therefore \frac{du}{d\theta} = -\frac{e}{l}.\sin\theta$$

$$\therefore \frac{d^2u}{d\theta^2} = -\frac{e}{l}\cos\theta$$

$$\therefore \frac{P}{h^2u^2} = u + \frac{d^2u}{d\theta^2} = \frac{1}{l} + \frac{e}{l}\cos\theta - \frac{e}{l}\cos\theta = \frac{1}{l}$$

$$\therefore P = \frac{h^2u^2}{l} = \frac{h^2}{l}.\frac{1}{r^2} = \frac{\mu}{r^2} \quad \left[\because \frac{h^2}{l} = \mu\right]$$

$$\boxed{\therefore P \alpha \frac{1}{r^2}}$$

5.4 Apses and apsidal distances

Definition

If there is a point A on a central orbit at which the velocity of the particle is perpendicular to the radius OA, then the point A is called an apse and the length OA is the apsidal distance.

Note : At an apse, the particle is moving at right angles to the radius vector.

We know that $\frac{1}{p^2} = u^2 + \left(\frac{du}{d\theta}\right)^2$ where $u = \frac{1}{r}$

At an apse, $p = r = \frac{1}{u}$. $\;\therefore$ At an apse, $\frac{du}{d\theta} = 0$

Given the law of force to the pole, find the orbit

Given the central acceleration P, we find the path. We use the equation.

$$u + \frac{d^2u}{d\theta^2} = \frac{P}{h^2u^2} \qquad \text{….. (1)}$$

To solve equation (1), we multiply both sides by $2\frac{du}{d\theta}$, we have

$$2u.\frac{du}{d\theta} + 2\frac{du}{d\theta}\cdot\frac{d^2u}{d\theta^2} = 2\frac{P}{h^2u^2}\cdot\frac{du}{d\theta}$$

i.e. $$\frac{d}{d\theta}(u)^2 + \frac{d}{d\theta}\left(\frac{du}{d\theta}\right)^2 = \frac{2P}{h^2u^2}\cdot\frac{du}{d\theta}$$

Integrating with respect to θ,

$$u^2 + \left(\frac{du}{d\theta}\right)^2 = \int \frac{2P}{h^2u^2}\,du + \text{constant} \qquad \text{…… (2)}$$

Problem 13

A particle moves with an acceleration $\mu\left[3au^4 - 2(a^2 - b^2)u^5\right]$ and is projected from an apse at a distance (a + b) with a velocity $\frac{\sqrt{\mu}}{a+b}$. Prove that the equation to its orbit is r = $a + b\cos\theta$.

Solution:

Given $P = \mu\left[3au^4 - 2\left(a^2 - b^2\right)u^5\right]$

The differential equation to the path is

$$u + \frac{d^2u}{d\theta^2} = \frac{P}{h^2u^2} = \frac{\mu}{h^2}\left[3au^2 - 2\left(a^2 - b^2\right)u^3\right] \qquad \text{…. (1)}$$

Multiplying (1) by $2\frac{du}{d\theta}$ and integrating with respect to θ we get

$$u^2 + \left(\frac{du}{d\theta}\right)^2 = \frac{2\mu}{h^2}\int\left[3au^2 - 2\left(a^2 - b^2\right)u^3\right]du + C$$

$$= \frac{2\mu}{h^2}\left[au^3 - 2\left(a^2 - b^2\right)\frac{u^4}{2}\right] + C \quad \ldots\ldots\ldots\ldots\ldots.. (2)$$

Now h = pv = constant = $p_o v_o$ where p_o and v_o are the initial values of p and v respectively.

Given $v_o = \frac{\sqrt{\mu}}{a+b}$ and p_o = a + b as the particle is projected from an apse

Hence $h = (a+b)\frac{\sqrt{\mu}}{a+b} = \sqrt{\mu}$ i.e. $\boxed{h^2 = \mu}$

So (2) becomes $u^2 + \left(\frac{du}{d\theta}\right)^2 = 2[au^3 - (a^2 - b^2)\frac{u^4}{2}] + c$ ….. (3)

Initially at the apse $\frac{du}{d\theta} = 0$ and $u = \frac{1}{a+b}$

Hence substituting these in (3), we have

$$\frac{1}{(a+b)^2} = 2\left[\frac{a}{(a+b)^3} - \frac{(a^2 - b^2)}{2(a+b)^4}\right] + C$$

$$= \frac{2a}{(a+b)^3} - \frac{(a-b)}{(a+b)^3} + C = \frac{1}{(a+b)^2} + C$$

$$\Rightarrow C = 0$$

(3) $\Rightarrow \left(\frac{du}{d\theta}\right)^2 = 2au^3 - \left(a^2 - b^2\right) u^4 - u^2$

$$\frac{du}{d\theta} = \sqrt{2au^3 - (a^2 - b^2)u^4 - u^2} = u\sqrt{2au - (a^2 - b^2)u^2 - 1} \quad \ldots (4)$$

i.e $\frac{du}{u\sqrt{2au - (a^2 - b^2)u^2 - 1}} = d\theta$

Put $u = \frac{1}{r}$ $\therefore du = -\frac{1}{r^2} dr$

$$-\frac{1}{r^2} \cdot r \frac{dr}{\sqrt{\frac{2a}{r} - \frac{\left(a^2 - b^2\right)}{r^2} - 1}} = d\theta$$

$$-\frac{dr}{\sqrt{2ar - (a^2 - b^2) - r^2}} = d\theta$$

i.e $\frac{-dr}{\sqrt{b^2 - (r-a)^2}} = d\theta$

Integrating, $\cos^{-1}\left(\frac{r-a}{b}\right) = \theta + \alpha$ …… (5) where α is constant.

If θ is measured from the apse line, r = a + b and θ = 0.

$$\cos^{-1}\left(\frac{a+b-a}{b}\right) = 0 + \alpha$$

i.e $\cos^{-1} 1 = \alpha$ $\therefore \alpha = 0$

Hence (5) becomes $\cos^{-1}\left(\frac{r-a}{b}\right) = \theta$

i.e $\frac{r-a}{b} = \cos\theta$

r = a + b $\cos\theta$

Problem 14

A particle moves with a central acceleration equal to μ $\div$ (distance) and is projected from an apse at a distance 'a' with a velocity equal to n times that which would be acquired in falling from infinity. Show that the other apsidal distance is $\dfrac{a}{\sqrt{n^2-1}}$

Solution:

"Velocity from infinity" means the velocity that acquired by the particle in falling with the given acceleration from infinity to the particular point given.

If x is the distance at time t from the centre in this motion, the equation is $\ddot{x} = -\dfrac{\mu}{x^5}$

Multiply by $2\dot{x}$ and integrate

$$\dot{x}^2 = -2\mu\int \frac{1}{x^5}\,dx + A = \frac{\mu}{2x^4} + A$$

Where $x = \infty$, $\dot{x} = 0$. Hence A = 0 and $\dot{x}^2 = \dfrac{\mu}{2x^4}$

When x = a, $\dot{x}^2 = \dfrac{\mu}{2x^4}$ and $\dot{x} = \sqrt{\dfrac{\mu}{2a^4}}$

Hence v_o = initial velocity of projection $= n\sqrt{\dfrac{\mu}{2a^4}} = \dfrac{n}{a^2}\sqrt{\dfrac{\mu}{2}}$

For the central orbit, P = $\dfrac{\mu}{r^5} = \mu\ u^5$

The differential equation of the path is

$$u+\frac{d^2u}{d\theta^2}=\frac{P}{h^2u^2}=\frac{\mu}{h^2}u^3$$

Multiplying (1) by $2\frac{du}{d\theta}$ and integrate with respect to θ,

$$u^2+(\frac{du}{d\theta})^2=\frac{2\mu}{h^2}\int u^3du+C=\frac{\mu}{2h^2}u^4+C \qquad \ldots\ldots (2)$$

Initial values are $p_o = a$, $v_o=\frac{n}{a^2}\sqrt{\frac{\mu}{2}}$

Hence h = $p_ov_o=\frac{n}{a}\sqrt{\frac{\mu}{2}}$ or $h^2=\frac{n^2\mu}{2a^2}$ i.e. $\frac{\mu}{2h^2}=\frac{a^2}{n^2}$

$$\therefore\ u^2+(\frac{du}{d\theta})^2=\frac{a^2u^4}{n^2}+C \qquad \ldots.. (3)$$

Initially at an apse, $\frac{du}{d\theta}=0$ and $u=\frac{1}{a}$

So from (3), $\frac{1}{a^2}=\frac{1}{n^2a^2}+C$ $\therefore$ $\boxed{C=\frac{1}{a^2}-\frac{1}{n^2a^2}}$

$$\therefore u^2+(\frac{du}{d\theta})^2=\frac{a^2u^4}{n^2}+\frac{1}{a^2}-\frac{1}{n^2a^2} \qquad \ldots. (4)$$

To get the apsidal distance put $\frac{du}{d\theta}=0$ in (4)

Hence $\frac{a^2u^4}{n^2}+\frac{1}{a^2}-\frac{1}{a^2n^2}-u^2=0$

i.e $a^4u^4+n^2-1-a^2n^2u^2=0$

or $a^4 u^4 - n^2 a^2 u^2 + (n^2 - 1) = 0$

i.e. $(a^2 u^2 - 1)\,[a^2 u^2 - (n^2 - 1)] = 0$

i.e. $a^2 u^2 = 1$ or $a^2 u^2 = n^2 - 1$

i.e au = 1 or au = $\sqrt{n^2 - 1}$

$u = \dfrac{1}{a}$ gives the point of projection

$\therefore$ apsidal distance is $u = \dfrac{\sqrt{n^2 - 1}}{a}$ i.e. $r = \dfrac{a}{\sqrt{n^2 - 1}}$

Problem 15

A particle is moving with central acceleration $\mu\left(r^5 - c^4 r\right)$ being projected from an apse at a distance C with velocity $C^3\sqrt{\dfrac{2\mu}{3}}$, Show that its path is the curve $x^4 + y^4 = c^4$

Solution:

Differential equation of the path is

$$\frac{p}{h^2 r^2} = u + \frac{d^2 u}{d\theta^2} \qquad \ldots\ldots\ldots\ldots (1)$$

Given, P = $\mu\left(r^5 - c^4 r\right) = \mu\left(\dfrac{1}{u^5} - \dfrac{c^4}{u}\right)$

$$\therefore \frac{\mu}{h^2}\left(\frac{1}{u^7} - \frac{c^4}{u^3}\right) = u + \frac{d^2 u}{d\theta^2}$$

$$\therefore h^2\left(\frac{d^2 u}{d\theta^2} + u\right) = \mu\left(u^{-7} - c^4 \cdot u^{-3}\right)$$

Multiply by $2\dfrac{du}{d\theta}$ and integrate,

$$v^2 = h^2\left[\left(\frac{du}{d\theta}\right)^2 + u^2\right] = 2\mu\left[-\frac{1}{6u^6} + \frac{c^4}{2u^2}\right] + c_1 \quad \text{..... (2)}$$

Initially, r = c, ie. $u = \dfrac{1}{c},\ v = c^3\sqrt{\dfrac{2\mu}{3}},\ \dfrac{du}{d\theta} = 0$

$$\therefore c^6\left(\frac{2\mu}{3}\right) = h^2\left[0 + \frac{1}{c^2}\right] = 2\mu\left[-\frac{1}{6}c^6 + \frac{c^6}{2}\right] + c_1$$

$$\boxed{\therefore h^2 = \frac{2}{3}\mu c^8} \quad , \quad \boxed{c_1 = 0}$$

$$(2) \Rightarrow \frac{2}{3}\mu c^8\left[\left(\frac{du}{d\theta}\right)^2 + u^2\right] = 2\mu\left[-\frac{1}{6u^6} + \frac{c^4}{2u^2}\right]$$

$$\therefore \frac{c^8}{3}\left[\left(\frac{du}{d\theta}\right)^2 + u^2\right] = \frac{-1}{6u^6} + \frac{c^4}{2u^2}$$

$$\therefore c^8\left(\frac{du}{d\theta}\right)^2 = 3\left[\frac{-1}{6u^6} + \frac{c^4}{2u^2}\right] - c^8u^2$$

$$= \frac{1}{u^6}\left[-\frac{1}{2} - \left(c^4u^4 - \frac{3}{4}\right)^2 + \frac{9}{16}\right]$$

$$= \frac{1}{u^6}\left[\left(\frac{1}{4}\right)^2 - \left(c^4u^4 - \frac{3}{4}\right)^2\right]$$

$$\therefore c^4 \cdot \frac{du}{d\theta} = \pm \frac{1}{4u^3}\left[\sqrt{1-\left(4c^4u^4-3\right)^2}\right]$$

$$\therefore 4u^3c^4du = -\sqrt{1-\left(4c^4u^4-3\right)^2}\,d\theta$$

ie) $-\dfrac{4c^4u^3du}{\sqrt{1-\left(4c^4u^4-3\right)^2}} = d\theta$

$$\therefore \int -\frac{16c^4u^3du}{\sqrt{1-\left(4c^4u^4-3\right)^2}} = \int 4d\theta\,.$$

$$\therefore \cos^{-1}\left(4c^4u^4-3\right) = 4\theta + c_2 \qquad \ldots\ldots (3)$$

Initially, $u = \dfrac{1}{c},\ \theta = 0 \ \Rightarrow c_2 = 0$

$$\therefore \cos^{-1}\left(4c^4u^4-3\right) = 4\theta$$

$$\therefore 4c^4u^4 - 3 = \cos 4\theta$$

$$\therefore 4c^4 = r^4(3+\cos 4\theta) = r^4(3+2\cos^2 2\theta - 1)$$

$$= r^4(2+2\cos^2 2\theta) = r^4[2+2(2\cos^2\theta-1)^2]$$

$$= r^4[2+2(4\cos^4\theta - 4\cos^2\theta + 1)]$$

$$= r^4\left[4+4\left(2\cos^4\theta - 2\cos^2\theta\right)\right]$$

$$= 4r^4[1+2\cos^4\theta - 2\cos^2\theta]$$

$$= 4r^4[\cos^4\theta + (\cos^4\theta - 2\cos^2\theta + 1)]$$

$$= 4r^4[\cos^4\theta + (1-\cos^2\theta)^2]$$

$$4c^4 = 4r^4[\cos^4\theta + \sin^4\theta]$$

$$= 4[(r\cos\theta)^4 + (r\sin\theta)^4] = 4[x^4 + y^4]$$

$$\boxed{\therefore c^4 = x^4 + y^4}$$ where $x = r\cos\theta, y = r\sin\theta$

Problem 16

In a central orbit the force is $\mu\ u^3\left(3 + 2a^2u^2\right)$; if the particle be projected at a distance 'a' with a velocity $\sqrt{5\mu/a^2}$ in a direction making an angle $\tan^{-1}\left(1/2\right)$ with the radius, show that the equation to the path is r = a tanθ.

Solution:

The differential eqn. of the path is

$$\frac{d^2u}{d\theta^2} + u = \frac{p}{h^2u^2} = \frac{\mu\ u^3\left(3 + 2a^2u^2\right)}{h^2u^3}$$

$$\therefore h^2\left(\frac{d^2u}{d\theta^2} + u\right) = \mu\left(3u + 2a^2u^3\right)$$

Multiply by 2 $\frac{du}{d\theta}$ and integrating,

$$v^2 = a^2\left[\left(\frac{du}{d\theta}\right)^2 + u^2\right] = \mu\left[3u^2 + a^2u^4\right] + C \qquad \text{\_\_\_\_\_\_\_\_(1)}$$

Also, p = r sin ϕ

$\therefore$ Initially, p_o = a sin ϕ_0

Now, $\phi_0 = \tan^{-1}\left(\frac{1}{2}\right) \Rightarrow \tan\phi_0 = \frac{1}{2}$

$\therefore \sin\phi_0 = \frac{1}{\sqrt{5}}$

$\therefore po = a\sin\phi_0 = \frac{a}{\sqrt{5}}$ ____________(2)

$$\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2 = u^2 + \left(\frac{du}{d\theta}\right)^2$$

Initially, $\left(\frac{du}{d\theta}\right)^2 + u^2 = \frac{1}{po^2} = \frac{5}{a^2}$

Also, initially, $v = \sqrt{\frac{5\mu}{a^2}}$ given.

$$\therefore (1) \Rightarrow \mu\left[\left(\frac{du}{d\theta}\right)^2 + u^2\right] = \mu\left[3u^2 + a^2u^4\right] + \frac{\mu}{a^2}$$

i.e. $\left(\frac{du}{d\theta}\right)^2 = 2u^2 + a^2u^4 + \frac{1}{a^2} = \frac{2a^2u^2 + a^4u^4 + 1}{a^2}$

$$\therefore \left(\frac{du}{d\theta}\right)^2 = \frac{a^4u^4 + 2a^2u^2 + 1}{a^2} = \left(\frac{a^2u^2 + 1}{a}\right)^2$$

$$\therefore \frac{du}{d\theta} = \pm\left(\frac{a^2u^2 + 1}{a}\right)$$

i.e. $\int -\frac{a\,du}{a^2u^2 + 1} = \int d\theta$

$\therefore \cot^{-1}(au) = \theta + c_1$.

Initially, u = $\frac{1}{a}, \theta = \frac{\pi}{4}$ $\therefore c_1 = 0$

$\therefore \cot^{-1}(au) = \theta$. $\therefore au = \cot\theta$

$\therefore \frac{a}{r} = \frac{1}{\tan\theta}$ $\therefore$ $\boxed{r = a\tan\theta}$

Problem 17

A particle is projected from an apse at a distance 'a' with a velocity from infinity, the acceleration being $\mu\, u^7$ show that the equation to its path is $r^2 = a^2 \cos 2\theta$

Solution:

Eqn. of motion is, force = - ma

$$\therefore \mu\ u^7 = -\frac{d^2x}{dt^2} = -\frac{d}{dt}\left(\frac{dx}{dt}\right) = -\frac{d\left(\frac{dx}{dt}\right)}{dx}.\frac{dx}{dt}$$

We know $v = \dfrac{dx}{dt} \Rightarrow$ $\boxed{\dfrac{\mu}{x^7} = -v\dfrac{dv}{dx}}$

$$\therefore \int_0^v 2v dv = \int_{x=\infty}^{a} -2\mu x^{-7} dx$$

$$\therefore v^2 = -2\mu\left[\frac{x^{-6}}{-6}\right]_0^a = 2\frac{\mu\ a^{-6}}{6} = \frac{\mu}{3\ a^6}$$

Now, $u + \dfrac{d^2u}{d\theta^2} = \dfrac{P}{h^2u^2}$

$$\therefore h^2\left[u + \frac{d^2u}{d\theta^2}\right] = \frac{\mu\ u^7}{u^2} = \mu\ u^5$$

Multiply by $2\dfrac{du}{d\theta}$; and integrating,

$$\therefore h^2\left[2u\frac{du}{d\theta} + 2\frac{du}{d\theta}.\frac{d^2u}{d\theta^2}\right] = 2\mu\ u^5\frac{du}{d\theta}$$

$$\therefore h^2\left[u^2 + \left(\frac{du}{d\theta}\right)^2\right] = 2\frac{\mu.u^6}{6} + C.$$

i.e. $\dfrac{h^2}{p^2} = \dfrac{\mu\ u^6}{3} + C.$

Initially, $v = V, u = \dfrac{1}{a}$ Also at an apse $\dfrac{du}{d\theta} = 0$

$$\therefore V^2 = h^2\left[\frac{1}{a^2}\right] = \frac{\mu}{3}.\frac{1}{a^6} + C \qquad \text{\_\_\_\_\_\_\_\_\_\_\_\_}(2)$$

i.e. $\dfrac{\mu}{3a^6} = \dfrac{\mu}{3a^6} + C \Rightarrow 0 = C$

(2) $\Rightarrow$ Also, $\boxed{h^2 = \dfrac{\mu}{3}\dfrac{a^2}{a^6} = \dfrac{\mu}{3a^4}}$

$$\therefore \frac{\mu}{3a^4}\left[u^2 + \left(\frac{du}{d\theta}\right)^2\right] = \frac{\mu\ u^6}{3}$$

$$\therefore \left(\frac{du}{d\theta}\right)^2 + u^2 = a^4u^6$$

$$\therefore \left(\frac{du}{d\theta}\right)^2 = a^4u^6 - u^2$$

Also, u = $\dfrac{1}{r}$, $\dfrac{du}{d\theta} = -\dfrac{1}{r^2}\dfrac{dr}{d\theta}$

$$\therefore \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2 = \frac{a^4}{r^6} - \frac{1}{r^2} = \frac{a^4 - r^4}{r^6}, \frac{dr}{d\theta} = \pm\frac{\sqrt{a^4 - r^4}}{r}$$

$$\therefore \frac{-r\ dr}{\sqrt{a^4 - r^4}} = d\theta$$

Put z = r^2 $\therefore dz = 2rdr$

$$\therefore \frac{-dz}{\sqrt{a^4 - z^2}} = 2d\theta,$$

$$\therefore \int -\frac{dz}{\sqrt{(a^2)^2 - z^2}} = 2\int d\theta = 2\theta$$

i.e. $\cos^{-1}\left(\dfrac{z}{a^2}\right) = 2\theta + C_1$

Initially, r=a, i.e. z=r$^2 = a^2; \theta = 0 \Rightarrow C_1 = 0$

$$\therefore \cos^{-1}\left(\frac{z}{a^2}\right) = 2\theta \Rightarrow \frac{z}{a^2} = \cos 2\theta$$

$$\text{i.e.) } \frac{r^2}{a^2} = \cos 2\theta$$

$$\therefore \boxed{r^2 = a^2 \cos 2\theta}$$

5.5 Inverse Square Law

Newton's Law of Attraction

The mutual attraction between two particles of masses m_1 and m_2 placed at a distance 'r' apart is a force of magnitude $\gamma \frac{m_1 m_2}{r^2}$ where γ is a constant, known as the constant of gravitation.

Problem 18

A particle moves in a path so that its acceleration is always directed to a fixed point and is equal to $\dfrac{\mu}{(distance)^2}$; Show that its path is a conic section and distinguish between the three cases that arise .

Solution:

Given $P = \frac{\mu}{r^2}$.

The (p, r) equation to the path is $\frac{h^2}{p^3} \cdot \frac{dp}{dr} = P = \frac{\mu}{r^2}$ ……. (1)

i. e. $h^2 \frac{dp}{p^3} = \mu \frac{dr}{r^2}$

Integrate, $\frac{h^2}{-2p^2} = -\frac{\mu}{r} +$ constant

$\frac{h^2}{p2} = \frac{2\mu}{r} + C$ ……. (2)

We know (p, r) equation of a parabola is $p^2 = ar$

(p, r) equation of an ellipse is $\frac{b^2}{p^2} = \frac{2a}{r} - 1$

(p, r) equation of a hyperbola is $\frac{b^2}{p^2} = \frac{2a}{r} + 1$

Comparing these equations with equation (2)

We get (2) is a parabola if C = 0

(2) is an ellipse if C is negative

(2) is a hyperbola if C is positive

Hence (2) always represents a conic section

Since h = pv where v is the velocity in the orbit at any point P distant r from the pole, equation (2) becomes

$$v^2 = \frac{2\mu}{r} + C$$

$$v^2 - \frac{2\mu}{r} = C \qquad \text{........ (4)}$$

Now, C is zero, negative or positive according as v^2 is equal to, less than or greater than $\frac{2\mu}{r}$.

Hence the path is a parabola, an ellipse or a hyperbola according as $v^2 =, < or > \frac{2\mu}{r}$.

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